Math 554, Final Exam, Summer 2005 Solution

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc; although, by using enough paper, you can do the problems in any order that suits you.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**. Otherwise, get your grade from VIP.

There are 13 problems. Problems 1 through 9 are worth 8 points each. Problems 10 through 13 are worth 7 points each. The exam is worth a total of 100 points.

I will post the solutions on my website shortly after the class is finished.

1. **Let $f\colon E \to \mathbb{R}$ be a function which is defined on a subset E of $\mathbb{R}$. Define $\lim\limits_{x\to p} f(x) = L$. Use complete sentences. (Be sure to tell me what kind of a thing p is, and what kind of a thing L is.)**

Let $f\colon E \to \mathbb{R}$ be a function which is defined on a subset E of $\mathbb{R}$. Assume that p is a limit point of E and L is a real number. We say that $\lim\limits_{x\to p} f(x) = L$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $|x-p| < \delta$, $x \neq p$, and $x \in E$, then $|f(x) - L| \le \varepsilon$.

2. **STATE either version of the Bolzano-Weierstrass Theorem.**

(version 1.) Every bounded infinite set of real numbers has a limit point in $\mathbb{R}$.

(version 2.) Every bounded sequence of real numbers has a convergent subsequence.

3. **PROVE either version of the Bolzano-Weierstrass Theorem.**

Proof of version 1.

Let S be a bounded infinite subset of $\mathbb{R}$, and let I be a finite closed interval which contains S. Cut I in half. At least one of the resulting two closed subintervals of I contains infinitely many elements of S. Call this interval I_1. Continue in this manner to build the closed interval I_n, for each natural number n, with the length of I_n equal to $1/2^n$ times the length of I and I_n contains infinitely many elements of S. The nested interval property of $\mathbb{R}$ tells us that the intersection $\bigcap\limits_{n=1}^{\infty} I_n$ is non-empty. Let p be an element of $\bigcap\limits_{n=1}^{\infty} I_n$. We will show that p is a limit point of S. Given $\varepsilon > 0$, there exists n large enough that the length of I_n is less than ε. We know that $p \in I_n$. It follows that $I_n \subseteq N_\varepsilon(p)$. Furthermore, there is at least one element q of S with $q \neq p$ and $q \in N_\varepsilon(p)$; since $I_n \cap S$ is infinite.

Proof of version 2.

Let $\{a_n\}$ be a bounded sequence of real numbers. There are two cases to consider depending upon the cardinality of the set $\{a_n \mid n \in \mathbb{N}\}$.
Case 1: $\{a_n \mid n \in \mathbb{N}\}$ **is finite.** In this case, it is clear that some subsequence of $\{a_n\}$ is constant.
Case 2: $\{a_n \mid n \in \mathbb{N}\}$ **is infinite.** We apply version 1 of the Bolzano-Weierstrass Theorem. The set $\{a_n \mid n \in \mathbb{N}\}$ has a limit point p. Pick n_1 with $|a_{n_1} - p| < 1$. Pick $n_2 > n_1$ with $|a_{n_1} - p| < \frac{1}{2}$. Every open neighborhood of p contains infinitely many elements of $\{a_n \mid n \in \mathbb{N}\}$. We continue in this manner to pick $n_k > n_{k-1}$ with $|a_{n_k} - p| < \frac{1}{k}$. We see that the subsequence $\{a_{n_k}\}$ of the sequence $\{a_n\}$ converges to p.

4. **Define** *Cauchy sequence*. **Use complete sentences.**

The sequence $\{a_n\}$ is a a *Cauchy sequence* if for all $\varepsilon > 0$, there exists n_0 such that whenever $n, m > n_0$, then $|a_n - a_m| < \varepsilon$.

5. **PROVE that every Cauchy sequence converges.**

Let $\{a_n\}$ be a Cauchy sequence. It is easy to see that $\{a_n\}$ is bounded. Indeed, there exists n_0 with $|a_n - a_{n_0}| < 1$ for all $n > n_0$. In this case, $|a_n| \leq M = \max\{|a_{n_0}| + 1, |a_1|, \dots, |a_{n_0-1}|\}$ for all n. Version 2 of the Bolzano-Weierstrass Theorem guarantees the existence of a convergent subsequence $\{a_{n_k}\}$ of $\{a_n\}$. Let a be the limit of the subsequence $\{a_{n_k}\}$. We will prove that the entire sequence $\{a_n\}$ converges to a. Let $\varepsilon > 0$ be fixed, but arbitrary. The subsequence $\{a_{n_k}\}$ converges to a; so, there exists k_0 such that, whenever $k \geq k_0$, then $|a_{n_k} - a| < \frac{\varepsilon}{2}$. The sequence $\{a_n\}$ is a Cauchy sequence; so, there exists n_1 such that, whenever $n, m \geq n_1$, then $|a_n - a_m| \leq \frac{\varepsilon}{2}$. Pick $n_0 \geq \max\{n_1, n_{k_0}\}$. If $n > n_0$, then we may choose k with $k > k_0$ and $n_k > n_0$. We now have:

$$|a_n - a| \leq |a_n - a_{n_k}| + |a_{n_k} - a| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

6. **Let I be an interval and $f \colon I \to \mathbb{R}$ be a function which is differentiable at the point p of I. PROVE that f is continuous at p.**

The point p is a limit point of I; so it suffices to show that $\lim\limits_{x \to p} f(x) = f(p)$. The hypothesis tells us that $\lim\limits_{x \to p} \frac{f(x) - f(p)}{x - p}$ exists and is equal to $f'(p)$. It is clear that $\lim\limits_{x \to p} x - p$ exists and is equal to 0. We proved that the limit of a product is the product of the limits provided the individual limits exist and are finite. We conclude that

$$\lim_{x \to p} f(x) - f(p) = \lim_{x \to p} \frac{f(x) - f(p)}{x - p} \cdot \lim_{x \to p} x - p = f'(p) \cdot 0 = 0.$$

It follows that $\lim_{x \to p} f(x) = f(p)$, and f is continuous at p .

7. **Let f be a continuous function from the closed interval $[a, b]$ to $\mathbb{R}$. Let $\varepsilon > 0$ be fixed. Prove that there exists $\delta > 0$ such that: whenever x and y are in $[a, b]$ with $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. (Notice that you are supposed to prove that one δ works for every x and y.)**

The function f is continuous on $[a, b]$, so for each point $p \in [a, b]$ there exists "a δ depending on p", call it $\delta_p > 0$, such that whenever

$$x \in [a, b] \text{ with } |x - p| < \delta_p \implies |f(x) - f(p)| < \tfrac{\varepsilon}{2}. \tag{3}$$

Let $\mathcal{N} = \{N_{\frac{\delta_p}{2}}(p) \mid p \in [a, b]\}$. We see that $\mathcal{N}$ is an open cover of the compact set $[a, b]$. Thus, there exists a finite set of points $p_1, \dots, p_\ell$ in $[a, b]$ so that $\{N_{\frac{\delta_{p_1}}{2}}(p_1), \dots, N_{\frac{\delta_{p_\ell}}{2}}(p_\ell)\}$ covers $[a, b]$. Let $\delta = \min\{\frac{\delta_{p_1}}{2}, \dots \frac{\delta_{p_\ell}}{2}\}$. The number δ is the minimum of a FINITE set of POSITIVE numbers, so $\delta > 0$. I claim that this one δ works for ALL x and y . Suppose, x, y are in $[a, b]$ with $|x - y| < \delta$. The set $\{N_{\frac{\delta_{p_1}}{2}}(p_1), \dots, N_{\frac{\delta_{p_\ell}}{2}}(p_\ell)\}$ covers $[a, b]$, so there is an index i , with $1 \le i \le \ell$ with $|x - p_i| < \frac{\delta_{p_i}}{2}$. It follows that

$$|y - p_i| = |y - x + x - p_i| \le |y - x| + |x - p_i| \le \delta + \frac{\delta_{p_i}}{2} \le \frac{\delta_{p_i}}{2} + \frac{\delta_{p_i}}{2} = \delta_{p_i}.$$

Use (3) twice at $p = p_i$ to see

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f(p_i) + f(p_i) - f(y)| \le |f(x) - f(p_i)| + |f(p_i) - f(y)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

8. **Give an example of a bounded infinite closed set that does not contain any intervals. Explain thoroughly.**

Let $K = \{0\} \cup \{\frac{1}{n} \mid n = 1, 2, 3, \dots\}$. We see that K is bounded below by 0 and above by 1 . We see that the complement of K , which is

$$(-\infty, 0) \cup \left(\bigcup_{n \in \mathbb{N}} (\tfrac{1}{n+1}, \tfrac{1}{n}) \right) \cup (1, \infty),$$

is a union of open intervals. Every open interval is an open set. The union of open sets is an open set. Thus, the complement of K is an open set; hence, K is a closed set. The set K is obviously infinite. The set K contains no intervals.

9. **Let A be an index set. For each index a in A, let F_a be a closed subset of $\mathbb{R}$. Is the union $\bigcup_{a\in A} F_a$ always closed? If yes, prove the claim. If no, give a counterexample.**

NO. Let $A=\mathbb{N}$ and for each $n\in\mathbb{N}$, let $F_n=[\frac{1}{n},1]$. We see that each F_n is closed, but the union $\bigcup_{n\in\mathbb{N}} F_n=(0,1]$, which is not closed. (Zero is a limit point of the union, but 0 is not in the union.)

10. **Let f and g be functions from the subset E of $\mathbb{R}$ to $\mathbb{R}$, and let p be a limit point of $\mathbb{R}$. Suppose that $\lim_{x\to p} f(x)$ exists and equals A. Suppose, also, that $\lim_{x\to p} g(x)$ exists and equals B. Prove $\lim_{x\to p} f(x)g(x)$ exists and equals AB.**

Let $\varepsilon>0$ be arbitrary, but fixed. We see that

$$|f(x)g(x)-AB|=|f(x)g(x)-Ag(x)+Ag(x)-AB|\le|f(x)-A||g(x)|+|A||g(x)-B|.$$

• We are told that $\lim_{x\to p} g(x)=B$; hence, there exists $\delta_1>0$ such that, whenever $x\in E$, $x\neq p$, and $|x-p|<\delta_1$, then $|g(x)-B|<1$. For such x, we have

$$|g(x)|=|g(x)-B+B|\le|g(x)-B|+|B|<|B|+1.$$

In other words,

$$x\in E,\ x\neq p,\ |x-p|<\delta_1 \implies |g(x)|<|B|+1 \tag{1}$$

• We are told that $\lim_{x\to p} g(x)=B$; hence, there exists $\delta_2>0$ such that

$$x\in E,\ x\neq p,\ |x-p|<\delta_2 \implies |g(x)-B|<\frac{\varepsilon}{2(|A|+1)} \tag{2}$$

• We are told that $\lim_{x\to p} f(x)=A$; hence, there exists $\delta_3>0$ such that

$$x\in E,\ x\neq p,\ |x-p|<\delta_3 \implies |f(x)-A|<\frac{\varepsilon}{2(|B|+1)} \tag{3}$$

Now, we take $\delta=\min\{\delta_1,\delta_2,\delta_3\}$. We also take an arbitrary $x\in E$ with $x\neq p$ and $|x-p|<\delta$. We see from (1) that $|g(x)|<|B|+1$; from (2) that $|g(x)-B|<\frac{\varepsilon}{2(|A|+1)}$; and from (3) that $|f(x)-A|<\frac{\varepsilon}{2(|B|+1)}$. Therefore,

$$\begin{aligned}&|f(x)g(x)-AB|\le|f(x)-A||g(x)|+|A||g(x)-B|\\&<\frac{\varepsilon}{2(|B|+1)}(|B|+1)+|A|\frac{\varepsilon}{2(|A|+1)}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

11. **Let c_1 be an arbitrary element of the open interval $(0,1)$. For each $n \in \mathbb{N}$, let $c_{n+1} = \frac{1}{5}(c_n^2+2)$. Prove that the sequence $\{c_n\}$ is contractive.**

First we show, by induction, that $c_{n+1} \in (0,1)$ for all natural numbers n. We picked c_1 to satisfy this hypothesis. If $0 < c_n < 1$, then $0 < c_n^2 < 1$, $2 < c_n^2 + 2 < 3$, $\frac{2}{5} < \frac{1}{5}(c_n^2+2) < \frac{3}{5}$; and therefore, $0 < c_{n+1} < 1$.

We see that

$$|c_{n+1} - c_n| = |\tfrac{1}{5}(c_n^2+2) - \tfrac{1}{5}(c_{n-1}^2+2)| = \tfrac{1}{5}|c_n^2 - c_{n-1}^2| = \tfrac{1}{5}|c_n - c_{n-1}||c_n + c_{n-1}|$$

$$\leq \tfrac{1}{5}|c_n - c_{n-1}|(|c_n| + |c_{n-1}|) < \tfrac{1}{5}|c_n - c_{n-1}|(1+1).$$

We used the triangle inequality together with our earlier observation that every member of the sequence has absolute value less than 1. The constant $b = \frac{2}{5}$ is less than 1. We have shown that

$$|c_{n+1} - c_n| < \tfrac{2}{5}|c_n - c_{n-1}|;$$

therefore, the sequence $\{c_n\}$ is contractive.

12. **Consider the sequence $\{a_n\}$ with $a_1 = 4$, and for $n \in \mathbb{N}$, $a_{n+1} = \sqrt{2a_n+3}$. Prove that the sequence converges. Find the limit of the sequence.**

We first prove, by induction, that the sequence is bounded below by 3. We see that $3 < a_1$. We assume $3 < a_n$. It follows that $6 < 2a_n$, $9 < 2a_n + 3$, and $3 < \sqrt{2a_n+3} = a_{n+1}$.

Now we prove that the sequence is monotone decreasing. I know $3 < a_n$. It follows that $0 < (a_n - 3)(a_n + 1)$, or $0 < a_n^2 - 2a_n - 3$. It now follows that $2a_n + 3 < a_n^2$. All of the numbers in the previous inequality are positive; so, we see that $\sqrt{2a_n+3} < a_n$. In other words, $a_{n+1} < a_n$.

The sequence $\{a_n\}$ is monotone decreasing and bounded. We proved that every monotone bounded sequence has a real number limit. It follows that the sequence $\{a_n\}$ converges. Finally, now that we know that the sequence $\{a_n\}$ converges to some limit L, we see that L must satisfy $L = \sqrt{2L+3}$; thus, $L^2 = 2L+3$, or $L^2 - 2L - 3 = 0$, or $(L-3)(L+1) = 0$. So, L equals either 3 or -1. Every element of the sequence is positive; so, $L \neq -1$. We conclude that $L = 3$.

13. **Let $f(x) = \begin{cases} x^2 + x & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational.} \end{cases}$ Is f differentiable at 0? Prove your answer.**

We calculate

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x}.$$

We suspect that this limit might exist, and if so, we suspect that the limit is 1. Notice that if x is not zero and x is rational, then

$$|\tfrac{f(x)}{x} - 1| = |\tfrac{x^2+x}{x} - 1| = |(x+1) - 1| = |x|.$$

If x is irrational, then

$$|\tfrac{f(x)}{x} - 1| = |\tfrac{x}{x} - 1| = 0.$$

Let $\varepsilon > 0$ be arbitrary, but fixed. Take $\delta = \varepsilon$. Suppose $|x - 0| < \delta$. If x is irrational, then $|\frac{f(x)}{x} - 1| = 0$, and this is certainly less than ε. If x is not 0 and x is rational, then $|\frac{f(x)}{x} - 1| = |x| < \delta = \varepsilon$. We conclude that f is differentiable at 0, and $f'(0) = 1$.