Math 554, Exam 4, Summer 2005

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc; although, by using enough paper, you can do the problems in any order that suits you.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

There are 7 problems. Problem 1 is worth 8 points. Problems 2 through 7 are worth 7 points each. The exam is worth a total of 50 points.

If you would like, I will leave your graded exam outside my office door. You may pick it up any time before the next class. **If you are interested, be sure to tell me.**

I will post the solutions on my website shortly after the class is finished.

1. Let $E = \{q \in \mathbb{Q} \mid 0 \le q \le 1\}$. Is E a compact set? Explain thoroughly. (Recall that $\mathbb{Q}$ is the set of rational numbers.)

2. Define *compact*. **Use complete sentences.** Include everything that is necessary, but nothing more.

3. Define *continuous*. **Use complete sentences.** Include everything that is necessary, but nothing more.

4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} x^3 & \text{if } x \text{ is rational} \\ 5x-2 & \text{if } x \text{ is irrational.} \end{cases}$$

   Give a complete "ε - δ" proof that f is continuous at $p = 2$.

5. STATE the theorem which relates the limit of a function and the limit of various sequences. **Use complete sentences.** Include everything that is necessary, but nothing more.

6. Let K be a compact set and let $f\colon K \to \mathbb{R}$ be a continuous function. Prove that the image $f(K)$ is a compact set. (I want to see a complete proof.)

7. Let f be a continuous function from the closed interval $[a,b]$ to $\mathbb{R}$. Let $\varepsilon > 0$ be fixed. Prove that there exists $\delta > 0$ such that: whenever x and y are in $[a,b]$ with $|x-y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. (Notice that you are supposed to prove that one δ works for every x and y.)