Math 554, Exam 3, Summer 2005

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc; although, by using enough paper, you can do the problems in any order that suits you.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

There are 6 problems. Problems 1 through 2 are worth 9 points each. Problems 3 through 6 are worth 8 points each. The exam is worth a total of 50 points.

If you would like, I will leave your graded exam outside my office door. You may pick it up any time before the next class. **If you are interested, be sure to tell me.**

I will post the solutions on my website shortly after the class is finished.

1. For each natural number n, let
$$s_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}.$$
Prove that $\{s_n\}$ is a Cauchy sequence.

2. Let A be a set. For each $a \in A$, let U_a be an open subset of $\mathbb{R}$.
   (a) Does $\bigcup\limits_{a\in A} U_a$ have to be an open set? If yes, prove the statement? If no, give a counterexample.
   (b) Does $\bigcap\limits_{a\in A} U_a$ have to be an open set? If yes, prove the statement? If no, give a counterexample.

3. Define *the sequence converges.* **Use complete sentences.** Include everything that is necessary, but nothing more.

4. For each natural number $n \in \mathbb{N}$, let K_n be a closed set of the form $(-\infty, b_n)$ for some $b_n \in \mathbb{R}$. Assume $K_n \supseteq K_{n+1}$ for all n. Does $\bigcup\limits_{n=1}^{\infty} K_n$ have to be non-empty? If yes, prove the statement? If no, give a counterexample. **OOPS!. This problem is riddled with typos. The set $K_n = (-\infty, b_n)$ is not a closed set. Do the best job you can, then move on.**

5. Consider the sequence $\{a_n\}$ with $a_1 = 10$ and, for all $n \geq 2$, $a_n = \frac{1}{2}(a_{n-1} + \frac{7}{a_{n-1}})$.
   (a) Show that this sequence is bounded below by $\sqrt{7}$.
   (b) Show that the sequence is a decreasing sequence.

6. Consider the sequence $\{a_n\}$ with $a_1 = \frac{1}{4}$ and, for all $n \geq 2$, $a_n = \frac{1}{3}(1 - a_{n-1}^3)$.
   (a) Show that $0 < a_n < \frac{1}{3}$, for all n.
   (b) Prove that $\{a_n\}$ is a contractive sequence.