Math 554, Exam 1, Summer 2005 Solutions

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc; although, by using enough paper, you can do the problems in any order that suits you.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

There are 7 problems. Problems 1 through 5 are worth 5 points each. Problem 6 is worth 10 points. Problem 7 is worth 15 points. The exam is worth a total of 50 points.

If you would like, I will leave your graded exam outside my office door. You may pick it up any time before the next class. **If you are interested, be sure to tell me.**

I will post the solutions on my website shortly after the class is finished.

1. **Define** *upper bound*. **Use complete sentences. Include everything that is necessary, but nothing more.**

The real number u is an *upper bound* of the non-empty set of real numbers E if $e \le u$ for all $e \in E$.

2. **Define** *supremum*. **Use complete sentences. Include everything that is necessary, but nothing more.**

The real number α is the *supremum* of the non-empty set of real numbers E if α is an upper bound of E and if d is a real number with $d < \alpha$, then d is not an upper bound of E.

3. **State the least upper bound axiom of the real numbers. Use complete sentences. Include everything that is necessary, but nothing more.**

Every non-empty set of real numbers which is bounded from above has a supremum.

4. **State the Archimedian property of the real numbers. Use complete sentences. Include everything that is necessary, but nothing more.**

If x and y are real numbers with $x > 0$, then there exists a positive integer n with $y < nx$.

5. **Let S be the following set of order pairs:**

$$S = \{(a, b) \mid a, b \in \mathbb{N}\}.$$

Is S a countable set? If yes, exhibit a one-to-one and onto function $f\colon \mathbb{N} \to S$. If no, why not?

YES. I will first draw S:

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \dots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \dots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \dots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \dots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

I count the set S by counting down the diagonals as indicated:

$$\begin{array}{cccccc}
\overset{(1,1)}{1} & \overset{(1,2)}{2} & \overset{(1,3)}{4} & \overset{(1,4)}{7} & \overset{(1,5)}{11} & \dots \\
\overset{(2,1)}{3} & \overset{(2,2)}{5} & \overset{(2,3)}{8} & \overset{(2,4)}{12} & (2,5) & \dots \\
\overset{(3,1)}{6} & \overset{(3,2)}{9} & \overset{(3,3)}{13} & (3,4) & (3,5) & \dots \\
\overset{(4,1)}{10} & \overset{(4,2)}{14} & (4,3) & (4,4) & (4,5) & \dots \\
\overset{(5,1)}{15} & (5,2) & (5,3) & (5,4) & (5,5) & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

6. **Suppose $f\colon X \to Y$ and $g\colon Y \to X$ are functions and that $g \circ f$ is the identity function on X. (In other words, $g(f(x)) = x$ for all $x \in X$.)**
   (a) **Does the function f have to be one-to-one? If yes, prove it. If no, give a counter example.**

YES. Suppose x_1 and x_2 are in X with $f(x_1) = f(x_2)$. Apply g to both sides to see that $g(f(x_1)) = g(f(x_2))$. The hypotheses tells us that $g(f(x_1)) = x_1$ and $g(f(x_2)) = x_2$. We have shown that

$$f(x_1) = f(x_2) \implies x_1 = x_2.$$

   (b) **Does the function f have to be onto? If yes, prove it. If no, give a counter example.**

NO. Let $X = \{1\}$, $Y = \{1, 2\}$, $f(1) = 1$, $g(1) = 1$, and $g(2) = 1$. We see that $g(f(1)) = g(1) = 1$ and 1 is the only element in X; therefore, $g(f(x)) = x$ for all $x \in X$. We also see that Y is not onto.

7. **Suppose that A and B are non-empty sets of real numbers with $12 \le a \le 20$ for all $a \in A$ and $2 \le b \le 4$ for all $b \in B$. Let**

$$C = \{\tfrac{a}{b} \mid a \in A \textbf{ and } b \in B\}.$$

   (a) What is an upper bound for C? Prove your answer.

Let c be an element of C. So $c = \frac{a}{b}$ for some $a \in A$ and $b \in B$. We are told that $a \le 20$ and $2 \le b$. It follows that $\frac{1}{b} \le \frac{1}{2}$. All of the numbers involved are positive. We conclude that $c = a(\frac{1}{b}) \le 20(\frac{1}{2}) = 10$. We have shown that 10 is an upper bound for C.

(b) **Give a formula for** $\sup C$ **in terms of** $\sup A$**,** $\sup B$**,** $\inf A$**, and** $\inf B$**.**

$$\sup C = \frac{\sup A}{\inf B}$$

(c) **Prove your answer to (b).**

Let $\alpha = \sup A$, $\beta = \inf B$ and $\gamma = \sup C$.

We prove that $\gamma \le \frac{\alpha}{\beta}$**:** Let c be an element of C. So $c = \frac{a}{b}$ for some $a \in A$ and $b \in B$. We know that $a \le \sup A = \alpha$ and $\beta = \inf B \le b$. It follows that $\frac{1}{b} \le \frac{1}{\beta}$. All of the numbers involved are positive. We conclude that $c = a(\frac{1}{b}) \le \alpha(\frac{1}{\beta})$. We have shown that $\frac{\alpha}{\beta}$ is an upper bound for C; and therefore, $\gamma = \sup C \le \frac{\alpha}{\beta}$.

We prove that $\gamma < \frac{\alpha}{\beta}$ **is not possible.** Suppose $\gamma < \frac{\alpha}{\beta}$. All of the numbers involved are positive; so, $\beta\gamma < \alpha = \sup A$. It follows that there is an element $a \in A$ with $\beta\gamma < a$. Divide by a positive number to see that $\inf B = \beta < \frac{a}{\gamma}$. It follows that there is an element $b \in B$ with $b < \frac{a}{\gamma}$. Multiply by positive numbers to see $\gamma < \frac{a}{b}$. Thus, there is an element of C which is larger than $\sup C$. This is a contradiction; hence, our supposition $\gamma < \frac{\alpha}{\beta}$ is incorrect.

We showed that $\gamma \le \frac{\alpha}{\beta}$. We also showd that γ is NOT less than $\frac{\alpha}{\beta}$. The only remaining possibility is that $\gamma = \frac{\alpha}{\beta}$.