Math 554, Final Exam Summer 2004

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Take enough space for each problem. Turn in your solutions in the order: problem 1, problem 2, ... ; although, by using enough paper, you can do the problems in any order that suits you.

There are 16 problems. Problems 1, 2, 3, and 4 are worth 7 points each. Problems 5 through 16 are worth 6 points each. The exam is worth a total of 100 points.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**. Otherwise, get your course grade from VIP.

I will post the solutions on my website shortly after the class is finished.

1. **Define** *Cauchy sequence*. **Use complete sentences.**

The sequence $\{a_n\}$ is a a *Cauchy sequence* if for all $\varepsilon > 0$, there exists n_0 such that whenever $n, m > n_0$, then $|a_n - a_m| < \varepsilon$.

2. **Let $f\colon E \to \mathbb{R}$ be a function which is defined on a subset E of $\mathbb{R}$. Define $\lim\limits_{x \to p} f(x) = L$. Use complete sentences. (Be sure to tell me what kind of a thing p is, and what kind of a thing L is.)**

Let $f\colon E \to \mathbb{R}$ be a function which is defined on a subset E of $\mathbb{R}$. Assume that p is a limit point of E and L is a real number. We say that $\lim\limits_{x \to p} f(x) = L$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $|x - p| < \delta$, $x \neq p$, and $x \in E$, then $|f(x) - L| \le \varepsilon$.

3. **Define** *continuous*. **Use complete sentences.**

Let E be a subset of $\mathbb{R}$. The function $f\colon E \to \mathbb{R}$ is continuous at the point p of E, if, for all $\varepsilon > 0$, there exists $\delta > 0$, such that whenever $|x - p| < \delta$ and $x \in E$, then $|f(x) - f(p)| < \varepsilon$.

4. **STATE either version of the Bolzano-Weierstrass Theorem.**

(version 1.) Every bounded infinite set of real numbers has a limit point in $\mathbb{R}$.

(version 2.) Every bounded sequence of real numbers has a convergent subsequence.

5. **PROVE either version of the Bolzano-Weierstrass Theorem.**

Proof of version 1.

Let S be a bounded infinite subset of $\mathbb{R}$, and let I be a finite closed interval which contains S. Cut I in half. At least one of the resulting two closed subintervals of I contains infinitely many elements of S. Call this interval I_1. Continue in this manner to build the closed interval I_n, for each natural number n, with the length of I_n equal to $1/2^n$ times the length of I and I_n contains infinitely many elements of S. The nested interval property of $\mathbb{R}$ tells us that the intersection

$\bigcap_{n=1}^{\infty} I_n$ is non-empty. Let p be an element of $\bigcap_{n=1}^{\infty} I_n$. We will show that p is a limit point of S. Given $\varepsilon > 0$, there exists n large enough that the length of I_n is less than ε. We know that $p \in I_n$. It follows that $I_n \subseteq N_\varepsilon(p)$. Furthermore, there is at least one element q of S with $q \neq p$ and $q \in N_\varepsilon(p)$; since $I_n \cap S$ is infinite.

Proof of version 2.

Let $\{a_n\}$ be a bounded sequence of real numbers. There are two cases to consider depending upon the cardinality of the set $\{a_n \mid n \in \mathbb{N}\}$.
Case 1: $\{a_n \mid n \in \mathbb{N}\}$ is finite. In this case, it is clear that some subsequence of $\{a_n\}$ is constant.
Case 2: $\{a_n \mid n \in \mathbb{N}\}$ is infinite. We apply version 1 of the Bolzano-Weierstrass Theorem. The set $\{a_n \mid n \in \mathbb{N}\}$ has a limit point p. Pick n_1 with $|a_{n_1} - p| < 1$. Pick $n_2 > n_1$ with $|a_{n_1} - p| < \frac{1}{2}$. Every open neighborhood of p contains infinitely many elements of $\{a_n \mid n \in \mathbb{N}\}$. We continue in this manner to pick $n_k > n_{k-1}$ with $|a_{n_k} - p| < \frac{1}{k}$. We see that the subsequence $\{a_{n_k}\}$ of the sequence $\{a_n\}$ converges to p.

6. **PROVE that every Cauchy sequence converges.**

Let $\{a_n\}$ be a Cauchy sequence. It is easy to see that $\{a_n\}$ is bounded. Indeed, there exists n_0 with $|a_n - a_{n_0}| < 1$ for all $n > n_0$. In this case, $|a_n| \leq M = \max\{|a_{n_0}| + 1, |a_1|, \ldots, |a_{n_0-1}|\}$ for all n. Version 2 of the Bolzano-Weierstrass Theorem guarantees the existence of a convergent subsequence $\{a_{n_k}\}$ of $\{a_n\}$. Let a be the limit of the subsequence $\{a_{n_k}\}$. We will prove that the entire sequence $\{a_n\}$ converges to a. Let $\varepsilon > 0$ be fixed, but arbitrary. The subsequence $\{a_{n_k}\}$ converges to a; so, there exists k_0 such that, whenever $k \geq k_0$, then $|a_{n_k} - a| < \frac{\varepsilon}{2}$. The sequence $\{a_n\}$ is a Cauchy sequence; so, there exists n_1 such that, whenever $n, m \geq n_1$, then $|a_n - a_m| \leq \frac{\varepsilon}{2}$. Pick $n_0 \geq \max\{n_1, n_{k_0}\}$. If $n > n_0$, then we may choose k with $k > k_0$ and $n_k > n_0$. We now have:

$$|a_n - a| \leq |a_n - a_{n_k}| + |a_{n_k} - a| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

7. **Let E be a set which is not closed. PROVE that E is not compact by constructing an open cover of E which does not admit a finite subcover.**

The set E is not closed; so, some limit point p of E is not contained in E. We construct a sequence $\{p_n\}$ in E which converges to p and which has no other limit points. Pick $p_1 \in E$ with $|p_1 - p| < 1$. Once p_n has been found, pick p_{n+1} in E with $|p_{n+1} - p| < \frac{1}{2}|p_n - p|$. The fact that p is a limit point of E guarantees the existence of p_{n+1}. Let S be the set $\{p_n \mid n \in \mathbb{N}\}$. We have constructed p_n in a manner which guarantees that S is infinite and the only limit point of S is p. For each $x \in E$, x is not a limit point of S, so we may pick ε_x so that $N_{\varepsilon_x}(x)$ contains at most one element of S. Let $\mathcal{U} = \{N_{\varepsilon_x}(x) \mid x \in E\}$. It is clear that $\mathcal{U}$ is an open cover of E. It is also clear that no finite subset of $\mathcal{U}$ can cover E; since a finite subset of $\mathcal{U}$ can cover only a finite subset of the infinite set S.

8. **PROVE that the continuous image of a compact set is compact.**

Let K be a compact subset of $\mathbb{R}$ and let $f\colon K \to \mathbb{R}$ be a continuous function. Let $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ be an open cover of $f(K)$. For each point $p \in K$, the element $f(p)$ is in $f(K)$. The set $\mathcal{U}$ covers $f(K)$, so there is an index α_p such that $f(p)$ is in U_{α_p}. The function f is continuous at p; so there exists a $\delta_p > 0$ such that $f(N_{\delta_p}(p) \cap K) \subseteq U_{\alpha_p}$. We create such a neighborhood $N_{\delta_p}(p)$ for each $p \in K$. We see that $\mathcal{N} = \{N_{\delta_p}(p) \mid p \in K\}$ is an open cover of K. The set K is compact; consequently, there exist $p_1, \dots, p_n$ in K such that $N_{\delta_{p_1}}(p_1), \dots, N_{\delta_{p_n}}(p_n)$ cover K. It follows that $f(N_{\delta_{p_1}}(p_1) \cap K), \dots, f(N_{\delta_{p_n}}(p_n) \cap K)$ cover $f(K)$. But $f(N_{\delta_{p_i}}(p_i) \cap K) \subseteq U_{\alpha_{p_i}}$, for all i; therefore, $U_{\alpha_{p_1}}, \dots, U_{\alpha_{p_n}}$ covers $f(K)$.

9. **Let I be an interval and $f\colon I \to \mathbb{R}$ be a function which is differentiable at the point p of I. PROVE that f is continuous at p.**

The point p is a limit point of I; so it suffices to show that $\lim\limits_{x \to p} f(x) = f(p)$. The hypothesis tells us that $\lim\limits_{x \to p} \frac{f(x)-f(p)}{x-p}$ exists and is equal to $f'(p)$. It is clear that $\lim\limits_{x \to p} x - p$ exists and is equal to 0. We proved that the limit of a product is the product of the limits provided the individual limits exist and are finite. We conclude that

$$\lim_{x \to p} f(x) - f(p) = \lim_{x \to p} \frac{f(x) - f(p)}{x - p} \cdot \lim_{x \to p} x - p = f'(p) \cdot 0 = 0.$$

It follows that $\lim\limits_{x \to p} f(x) = f(p)$, and f is continuous at p.

10. **Let A and B be non-empty sets, and let $f\colon A \to B$ and $g\colon B \to A$ be functions. Suppose that $g \circ f$ is the identity function on A. (In other words, $g(f(a)) = a$ for all a in A.) Does f have to be onto? If yes, PROVE the result. If no, then give an EXAMPLE.**

$\boxed{\text{NO}}$. Let $A = \{1\}$, $B = \{1, 2\}$, $f\colon A \to B$ be the function which sends 1 to 1, and $g\colon B \to A$ be the function which sends $g(1) = g(2) = 1$. We see that $g(f(1)) = 1$, but f is not onto.

11. **Give an example of a countable set E and an open cover $\mathcal{U}$ of E which does not admit a finite subcover of E.**

Let E be the set $\{\frac{1}{n}\}$ and let $\mathcal{U} = \{U_n \mid n \in \mathbb{N}\}$, where $U_n = N_{\frac{1}{n} - \frac{1}{n+1}}(\frac{1}{n})$. It is clear that $\mathcal{U}$ is an open cover of E. A little arithmetic shows that $\frac{1}{n}$ is the only element of E which is in U_n; consequently, it is not possible to cover E with any finite subset of $\mathcal{U}$.

12. **Let $f(x) = \begin{cases} x^3 & \textbf{if } x \textbf{ is irrational} \\ 0 & \textbf{if } x \textbf{ is rational.} \end{cases}$ Does $f'(0)$ exist? PROVE your answer completely, using ε's and δ's.**

We show that $f'(0) = 0$. Let $\varepsilon > 0$ be fixed but arbitrary. Take $\delta = \sqrt{\varepsilon}$. We will prove that if $|x - 0| < \delta$ and $x \neq 0$, then $|\frac{f(x)-f(0)}{x-0}| < \varepsilon$. That is, we prove

$$(*) \qquad |x| < \delta, x \neq 0 \implies |\tfrac{f(x)}{x}| < \varepsilon.$$

There are two cases. If x is rational (and $x \neq 0$), then $f(x) = 0$ and $\frac{f(x)}{x} = \frac{0}{x} = 0$ and (*) holds. On the other hand, if x is irrational, then $f(x) = x^3$ and $\frac{f(x)}{x} = \frac{x^3}{x} = x^2$; consequently, if $|x| < \sqrt{\varepsilon}$, then $|x^2| < \varepsilon$ and (*) continues to hold.

13. **Let** $f(x) = \begin{cases} 5x - 3 & \textbf{if } x \leq 1 \\ 4 - 2x & \textbf{if } 1 < x. \end{cases}$ **Is** f **continuous at** $x = 1$**? PROVE your answer completely, using** ε**'s and** δ**'s.**

Let $\varepsilon > 0$ be fixed but arbitrary. Let $\delta = \frac{\varepsilon}{5}$. We prove that if $|x - 1| < \delta$, then $|f(x) - f(1)| < \varepsilon$. There are two cases. If $x < 1$, then

$$|f(x) - f(1)| = |5x - 3 - 2| = 5|x - 1| < 5\tfrac{\varepsilon}{5} = \varepsilon.$$

On the other hand, if $x \leq 1$, then

$$|f(x) - f(1)| = |4 - 2x - 2| = 2|x - 1| < 2\tfrac{\varepsilon}{5} < \varepsilon.$$

14. **For each integer** n**, let** I_n **be the open interval** $(\frac{1}{n}, 2 + \frac{1}{n})$**. Compute** $\bigcap\limits_{n=1}^{\infty} I_n$**.**

We see that $I_1 = (1, 3)$, $I_2 = (\frac{1}{2}, 2 + \frac{1}{2})$, etc. We conclude that the intersection is $\boxed{(1, 2]}$.

15. **Let** $a_1 \neq a_2$ **be real numbers. For** $n \geq 3$**, let** $a_n = \frac{2}{3}a_{n-1} + \frac{1}{3}a_{n-2}$**. PROVE that the sequence** $\{a_n\}$ **is a contractive sequence.**

We see that

$$\frac{|a_{n+2} - a_{n+1}|}{|a_{n+1} - a_n|} = \frac{|\frac{2}{3}a_{n+1} + \frac{1}{3}a_n - a_{n+1}|}{|a_{n+1} - a_n|} = \frac{|-\frac{1}{3}a_{n+1} + \frac{1}{3}a_n|}{|a_{n+1} - a_n|} = \frac{|-\frac{1}{3}||a_{n+1} - a_n|}{|a_{n+1} - a_n|} = \frac{1}{3}.$$

Thus, $|a_{n+2} - a_{n+1}| \leq \frac{1}{3}|a_{n+1} - a_n|$ for all n and the sequence $\{a_n\}$ is a contractive sequence.

16. **Let** $a_1 = \sqrt{2}$ **and for each integer** $n \geq 1$**, let** $a_{n+1} = \sqrt{2 + a_n}$**. PROVE that** $a_n \leq 2$ **for all** n**. PROVE that the sequence** $\{a_n\}$ **is a monotone increasing sequence.**

We prove $a_n \leq 2$ by induction on n. We see that $a_1 \leq 2$. By induction, we assume that $a_n \leq 2$. It follows that $2 + a_n \leq 4$ and $a_{n+1} = \sqrt{2 + a_n} \leq \sqrt{4} = 2$.

Now we prove that $a_n \leq a_{n+1}$. We just showed that $a_n - 2 \leq 0$. Multiply both sides by the positive number $a_n + 1$ to see that $(a_n - 2)(a_n + 1) \leq 0$. That is, $a_n^2 - a_n - 2 \leq 0$, or $a_n^2 \leq a_n + 2$. Take the square root (keep in mind that $a_n > 0$) to see that $a_n \leq \sqrt{a_n + 2} = a_{n+1}$.