Math 554, Exam 3, Summer 2004

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Take enough space for each problem. Turn in your solutions in the order: problem 1, problem 2, ... ; although, by using enough paper, you can do the problems in any order that suits you.

There are 9 problems. Problems 1 through 5 are worth 6 points each. Problems 6 through 9 are worth 5 points each. The exam is worth a total of 50 points.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will leave your exam outside my office door by noon tomorrow, you may pick it up any time between then and the next class.

I will post the solutions on my website shortly after the class is finished.

1. Define *Cauchy sequence*. Use complete sentences.

2. Define *limit point*. Use complete sentences.

3. For each natural number n, let U_n be an open set in $\mathbb{R}$. Is the intersection $\bigcap_{n=1}^{\infty} U_n$ always an open set? If yes, prove the result. If no, give a counterexample.

4. For each natural number n, let U_n be an open set in $\mathbb{R}$. Is the union $\bigcup_{n=1}^{\infty} U_n$ always an open set? If yes, prove the result. If no, give a counterexample.

5. State the theorem which characterizes the closed sets of $\mathbb{R}$ in terms of information about the limit points.

6. Prove the first version of the Bolzano-Weierstrass Theorem. That is, prove that every bounded infinite subset of $\mathbb{R}$ has a limit point.

7. Let $\{p_n\}$ be a bounded sequence of real numbers and let $p \in \mathbb{R}$ be such that every convergent subsequence of $\{p_n\}$ converges to p. Prove that the sequence $\{p_n\}$ converges to p.

8. Let a_1 be a real number in the open interval $(0,1)$. Define the sequence $\{a_n\}$ by $a_{n+1} = \frac{1}{5}(1 - a_n^3)$, for all $n \geq 1$. Prove that the sequence $\{a_n\}$ is a contractive sequence.

9. Let K be a closed non-empty subset of $\mathbb{R}$ and let x be an element of $\mathbb{R}$, with $x \notin K$. Prove that there exists at least one element y of K which is closest to x. In other words, if $z \in K$, then $|x - y| \leq |x - z|$.