Math 554, Exam 2 Solutions, Summer 2004

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.

There are 10 problems. Each problem is worth 5 points. The exam is worth a total of 50 points.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will leave your exam outside my office door by noon tomorrow, you may pick it up any time between then and the next class.

I will post the solutions on my website shortly after the class is finished.

1. **Define *supremum*. Use complete sentences.**

The real number α is the *supremum* of the set of real numbers E, if α is an upper bound of E, and no real number smaller than α is an upper bound of E.

2. **Define *limit point*. Use complete sentences.**

The real number p is a *limit point* of the set of real numbers E if, for all $\varepsilon > 0$, there exists $q \in E$ with $q \neq p$ and $|q - p| \leq \varepsilon$.

3. **State the least upper bound axiom.**

Every non-empty set of real numbers which is bounded from above has a supremum in $\mathbb{R}$.

4. **State either version of the Bolzano-Weierstrass Theorem.**

(version 1.) Every bounded infinite set of real numbers has a limit point in $\mathbb{R}$.

(version 2.) Every bounded sequence of real numbers has a convergent subsequence.

5. **Let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions with f one-to-one and g one-to-one, prove that the function $g \circ f\colon X \to Z$ is one-to-one.**

Take x_1 and x_2 in X with $(g \circ f)(x_1) = (g \circ f)(x_2)$. It follows that $g(f(x_1)) = g(f(x_2))$. The function g is one-to-one; so we know that $f(x_1) = f(x_2)$. The function f is one-to-one, so we know that $x_1 = x_2$.

6. **Give an example of a bounded set with exactly three limit points.**

The set

$$\{1 + \tfrac{1}{n} \mid n \in \mathbb{N}\} \cup \{2 + \tfrac{1}{n} \mid n \in \mathbb{N}\} \cup \{3 + \tfrac{1}{n} \mid n \in \mathbb{N}\}$$

has exactly 3 limit points; namely, 1, 2, and 3.

7. **For each natural number n, let I_n be the open interval $(0, \frac{1}{n})$. What is $\bigcap_{n=1}^{\infty} I_n$? Prove your answer.**

The intersection is empty. If x were an element of the intersection, then $0 < x < \frac{1}{n}$, for all positive integers n. There is no such x. (If one looks a a fixed positive number x, the Archimedian Property of $\mathbb{R}$ guarantees that there exists a natural number n with $\frac{1}{n} < x$.)

8. **Let $\{a_n\}$ be a sequence which converges to a and $\{b_n\}$ be a sequence which converges to b. Prove that the sequence $\{a_n b_n\}$ converges to ab.**

Let $\varepsilon > 0$ be arbitrary, but fixed.
• The sequence $\{a_n\}$ converges to a, so there exists n_1 such that if $n \geq n_1$, then $|a_n - a| \leq 1$. For such n, the Corollary to the triangle inequality tells us that

$$|a_n| - |a| \leq ||a_n| - |a|| \leq |a_n - a| \leq 1;$$

and therefore, $|a_n| \leq |a| + 1$.
• The sequence $\{b_n\}$ converges to b, so there exists n_2 such that if $n \geq n_2$, then $|b_n - b| \leq \frac{\varepsilon}{2(|a|+1)}$.
• The sequence $\{a_n\}$ converges to a, so there exists n_3 such that if $n \geq n_3$, then $|a_n - a| \leq \frac{\varepsilon}{2(|b|+1)}$.

Let n_0 be the maximum of the three integers n_1, n_2, and n_3. Take $n \geq n_0$. We know that $n \geq n_1$; and therefore,

$$|a_n| \leq |a| + 1. \tag{1}$$

We know that $n \geq n_2$; and therefore,

$$|b_n - b| \leq \frac{\varepsilon}{2(|a| + 1)}. \tag{2}$$

We know that $n \geq n_3$; and therefore,

$$|a_n - a| \leq \frac{\varepsilon}{2(|b| + 1)}. \tag{3}$$

The triangle inequality tells us that

$$|a_n b_n - ab| = |(a_n b_n - a_n b) + (a_n b - ab)| \leq |a_n b_n - a_n b| + |a_n b - ab| = |a_n||b_n - b| + |a_n - a||b|.$$

Use (1), (2) and (3) to see that

$$|a_n b_n - ab| \leq |a_n||b_n - b| + |a_n - a||b| \leq (|a| + 1)\frac{\varepsilon}{2(|a| + 1)} + \frac{\varepsilon}{2(|b| + 1)}|b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

9. **Let $\{b_n\}$ be a sequence which converges to b, with $b \neq 0$. Prove that the sequence $\{\frac{1}{b_n}\}$ converges to $\frac{1}{b}$.**

• The number $\frac{|b|}{2}$ is greater than zero. The sequence $\{b_n\}$ converges to b. Thus, there exists an integer n_1 such that if $n_1 \leq n$, then $|b_n - b| \leq \frac{|b|}{2}$. For such n, the Corollary to the triangle inequality tells us that

$$||b_n| - |b|| \leq |b_n - b| \leq \frac{|b|}{2}.$$

In this case,

$$-\frac{|b|}{2} \leq |b_n| - |b| \leq \frac{|b|}{2}.$$

Add $|b|$ to each term in this inequality to see that

$$+\frac{|b|}{2} \leq |b_n| \leq \frac{3|b|}{2}.$$

In particular, the left-most inequality gives an upper bound on $\frac{1}{|b_n|}$; namely,

$$\frac{1}{|b_n|} \leq \frac{2}{|b|}. \tag{4}$$

• The sequence $\{b_n\}$ converges to b. So, there exists an integer n_2 such that whenever $n \geq n_2$, then

$$|b - b_n| \leq \frac{\varepsilon|b|^2}{2}. \tag{5}$$

Take n_0 equal to the maximum of n_1 and n_2. If $n_0 \leq n$, then use (4) and (5) to see that

$$|\tfrac{1}{b_n} - \tfrac{1}{b}| = |\tfrac{b-b_n}{bb_n}| = |b - b_n|\tfrac{1}{|b|}\tfrac{1}{|b_n|} \leq \tfrac{\varepsilon|b|^2}{2}\tfrac{1}{|b|}\tfrac{2}{|b|} = \varepsilon.$$

10. **Let $\{p_n\}$ be a bounded sequence of real numbers and let $p \in \mathbb{R}$ be such that every convergent subsequence of $\{p_n\}$ converges to p. Prove that the sequence $\{p_n\}$ converges to p.**

Suppose that

(6) the sequence $\{p_n\}$ does NOT converge to p.

In this case, there exists $\varepsilon > 0$ such that

(7) for every $n_0 \in \mathbb{N}$ there exists $n > n_0$ such that $|p_n - p| \geq \varepsilon$.

Apply (7) to find $n_1 > 1$, with $|p_{n_1} - p| > \varepsilon$. Apply (7) to find $n_2 > n_1$, with $|p_{n_2} - p| > \varepsilon$. Apply (7) to find $n_3 > n_2$, with $|p_{n_3} - p| > \varepsilon$. Continue in this manner to construct a subsequence

$$p_{n_1}, p_{n_2}, p_{n_3}, \ldots \tag{8}$$

of the original sequence $\{p_n\}$ which never gets closer to p than ε. The Bolzano-Weierstrass Theorem version 2, see problem 4, guarantees that some subsequence of (8) converges. This subsequence of (8) does not converge to p because the subsequence never gets within ε of p. On the other hand, subsequence of (8) is also a subsequence of the original sequence $\{p_n\}$; and therefore, must converge to p by the original hypothesis. This is a contradiction. The original supposition (6) must be false. We conclude that the sequence $\{p_n\}$ does converge to p.