Math 554, Exam 1 Solutions, Summer 2004

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.

There are 7 problems. Problems 1 through 6 are worth 7 points each. Problem 7 is worth 8 points. The exam is worth a total of 50 points.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will leave your exam outside my office door by noon tomorrow, you may pick it up any time between then and the next class.

I will post the solutions on my website shortly after the class is finished.

1. **Define** ***upper bound*****.**

The real number u is an *upper bound* of the non-empty set of real numbers E if $e \le u$ for all $e \in E$.

2. **Define** ***supremum*****.**

The real number α is the *supremum* of the non-empty set of real numbers E if α is an upper bound of E and if d is a real number with $d < \alpha$, then d is not an upper bound of E.

3. **State the least upper bound axiom.**

Every non-empty set of real numbers which is bounded from above has a supremum.

4. **Let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions with f onto and g onto, prove that the function $g \circ f\colon X \to Z$ is onto.**

Take an arbitrary element $z \in Z$. The function g is onto, so there exists an element y in Y, with $g(y) = z$. The function f is onto, so there exists an element x in X with $f(x) = y$. We see that $(g \circ f)(x) = g(f(x)) = g(y) = z$.

5. **Let x and y be real numbers with $0 < x$. Prove that there exists a positive integer N with $Nx > y$. (I want you to give a complete proof of this result. I want more that its name. I want more than the statement that "we did this in class".)**

This is a proof by contradiction. Suppose that

(*) $$nx \le y \text{ for all positive integers } n.$$

Let $E = \{nx \mid n \in \mathbb{N}\}$. The supposition (*) tells us that y is an upper bound for E. The set E is non-empty because $x \in E$. The least upper bound axiom of $\mathbb{R}$ tells us that E has a supremum; which we call α. We know $0 < x$; so $\alpha - x < \alpha$. Thus, $\alpha - x$ is not an upper bound for E and $\alpha - x < nx$ for some $n \in \mathbb{N}$. We see that $\sup E = \alpha < (n+1)x \in E$. This is a contradiction. Supposition (*) must be false. Thus, there must exist $n \in \mathbb{N}$ with $y < nx$.

6. **Exhibit a one-to-one and onto function f from the open inteval $(3,4)$ to the open interval $(7,12)$.**

I look for a linear function with $f(3) = 7$ and $f(4) = 12$. That is, I want the line which passes through the points $(3,7)$ and $(4,12)$. The slope is 5 the line is $y - 7 = 5(x-3)$; or $f(x) = 5x - 8$. Notice that $f(3) = 7$, $f(4) = 12$. If $3 < x < 4$, then $5(3) < 5x < 5(4)$; so, $15 - 8 < 5x - 8 < 20 - 8 = 12$; that is $f(x) \in (7,12)$. Thus, $f\colon (3,4) \to (7,12)$ is a function. We see that f is one-to-one. If x_1 and x_2 are in $(3,4)$, with $f(x_1) = f(x_2)$, then $5x_1 - 8 = 5x_2 - 8$. Add 8 to both sides. Divide by 5 to see that $x_1 = x_2$. We see that f is onto. Take $y \in (7,12)$. Observe that $\frac{y+8}{5} \in (3,4)$, and $f(\frac{y+8}{5}) = y$.

7. **Let A and B be non-empty sets of real numbers. Suppose that $\inf A = 2$, $\sup A = 6$, $\inf B = 4$ and $\sup B = 30$. Let**

$$C = \{\tfrac{a}{b} \mid a \in A, \textbf{ and } b \in B\}.$$

What is $\inf C$? Give a complete proof that your answer is correct.

Let $\gamma = \inf C$. We show that $\gamma = \frac{1}{15}$. We first show that $\frac{1}{15}$ is a lower bound for C. Let c be an arbitrary element of C. So, $c = \frac{a}{b}$ for some $a \in A$ and $b \in B$. We know that $2 \le a$ and that $b \le 30$; hence, $\frac{1}{30} \le \frac{1}{b}$. We multiply these two inequalities to see that $\frac{2}{30} \le \frac{a}{b}$. Thus, $\frac{1}{15}$ is a lower bound for C and the greatest lower bound for C, namely γ satisfies $\frac{1}{15} \le \gamma$.

The rest of the argument is by contradiction:

(**) Suppose that $\frac{1}{15} < \gamma$.

It follows that $2 < 30\gamma$. But 2 is the greates lower bound of A; thus, 30γ is not a lower bound of A and there exists an element a of A with $a < 30\gamma$. We also have $\frac{a}{\gamma} < 30$. We know that 30 is the least upper bound for B. It follows that there exists an element $b \in B$ with $\frac{a}{\gamma} < b$. We have now found the element $c = \frac{a}{b} < \gamma$. This contradicts the fact that $c \in C$ and γ is a lower bound for C. Supposition (**) must be wrong.

We now know that $\frac{1}{15} \le \gamma$, but $\frac{1}{15}$ is not less than γ. The only remaining possibility is that $\frac{1}{15} = \gamma$