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Quiz for June 22, 2006

Let f and g be functions from the subset E of $\mathbb{R}$ to $\mathbb{R}$, and let p be a limit point of E. Suppose that $\lim\limits_{x\to p} f(x)$ exists and equals A. Suppose, also, that $\lim\limits_{x\to p} g(x)$ exists and equals B. Prove $\lim\limits_{x\to p} f(x)g(x)$ exists and equals AB. I expect a complete ε, δ proof.

ANSWER: Let $\varepsilon > 0$ be arbitrary, but fixed. We see that

$$|f(x)g(x)-AB| = |f(x)g(x)-Ag(x)+Ag(x)-AB| \le |f(x)-A||g(x)|+|A||g(x)-B|.$$

• We are told that $\lim\limits_{x\to p} g(x) = B$; hence, there exists $\delta_1 > 0$ such that, whenever $x \in E$, $x \neq p$, and $|x-p| < \delta_1$, then $|g(x)-B| < 1$. For such x, we have

$$|g(x)| = |g(x) - B + B| \le |g(x)-B| + |B| < |B| + 1.$$

In other words,

(1) $$x \in E,\ x \neq p,\ |x-p| < \delta_1 \implies |g(x)| < |B| + 1$$

• We are told that $\lim\limits_{x\to p} g(x) = B$; hence, there exists $\delta_2 > 0$ such that

(2) $$x \in E,\ x \neq p,\ |x-p| < \delta_2 \implies |g(x)-B| < \frac{\varepsilon}{2(|A|+1)}$$

• We are told that $\lim\limits_{x\to p} f(x) = A$; hence, there exists $\delta_3 > 0$ such that

(3) $$x \in E,\ x \neq p,\ |x-p| < \delta_3 \implies |f(x)-A| < \frac{\varepsilon}{2(|B|+1)}$$

Now, we take $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. We also take an arbitrary $x \in E$ with $x \neq p$ and $|x-p| < \delta$. We see from (1) that $|g(x)| < |B| + 1$; from (2) that $|g(x)-B| < \frac{\varepsilon}{2(|A|+1)}$; and from (3) that $|f(x)-A| < \frac{\varepsilon}{2(|B|+1)}$. Therefore,

$$|f(x)g(x)-AB| \le |f(x)-A||g(x)| + |A||g(x)-B|$$

$$< \frac{\varepsilon}{2(|B|+1)}(|B|+1) + |A|\frac{\varepsilon}{2(|A|+1)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$