Math 554 Summer 2000 Exam 3

① The real number p is a limit point of the set of real numbers E if for every $\varepsilon > 0$, there exists $x \in N_\varepsilon(p)$ with $x \in E$ and $x \neq p$.

② For each natural number n let I_n be a closed and bounded interval. If $I_n \supseteq I_{n+1}$ for all n, then $\bigcap_{n=1}^{\infty} I_n$ is not empty.

③ Let $\{a_n\}_{n=1}^{\infty}$ be a bounded monotone decreasing sequence of real numbers
The sequence is bounded so the set $E = \{a_n \mid n \in \mathbb{N}\}$ has an
infimum. Let $a = \inf E$. I claim that $a = \lim_{n\to\infty} a_n$. Given $\varepsilon>0$
$a-\varepsilon$ is too big to be a lower bound for E so $\exists\, a_{n_0} \in E$
with $a \le a_{n_0} < a - \varepsilon$. Our sequence is decreasing hence
$\forall n \ge n_0$ $a \le a_n \le a_{n_0} < a-\varepsilon$. We have shown that $|a - a_n| < \varepsilon$
for $n \ge n_0$. Thus we proved that the sequence $\{a_n\}$ converges to a.

④ Let S be a bounded infinite set of real numbers.
The set S is bounded so $S \subseteq [a,b]$ for some a and b.
If I chop S in half, one of the two halves must contain
infinitely many elements from S. Proceed in this manner to
obtain $I_0 \supseteq I_1 \supseteq \cdots$, with $I_0 = [a,b]$, the length
of $I_n = \frac{b-a}{2^n}$, and $I_n \cap S$ is infinite. The nested intervals
property tells me that $\bigcap_{n=1}^{\infty} I_n$ is not empty. Let $x \in \bigcap_{n=1}^{\infty} I_n$. I
claim that x is a limit point of S. This is essentially clear.
Given ε, take n large enough that the length of $I_n < \varepsilon$. We know
$x \in I_n$. If $y \in I_n$, then $|x-y| < \varepsilon$. We created I_n to have an
infinite number of elements of S. So take $y \in S \cap I_n$, $y \neq x$.
We have $|y - x| < \varepsilon$ and the proof is complete.

⑤ $\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{1 + \frac{1}{n} \mid n \in \mathbb{N}\} \cup \{2 + \frac{1}{n} \mid n \in \mathbb{N}\}$

The limit points are $0, 1, 2$

⑥ $E = \{0\}$ $\sup E = 0$ but E does not have any limit points.