Math 554 Summer 2000 Exam 2

(1) The real number p is the limit of the sequence $\{a_n\}$ if for all $\varepsilon > 0$ there exists n_0 (a positive integer) such that $|p - a_n| < \varepsilon$ whenever $n \geq n_0$.

(2) Let f be a function from the set A to the set B. The function f is one-to-one if $f(a_1) \neq f(a_2)$ in B whenever $a_1 \neq a_2$ in A.

(3) The set A is countable if there exists a one-to-one function from the set of natural numbers onto A.

(4) Define $f(x) = 3x - 1$. Notice that if $0 < x < 1$, then $0 < 3x < 3$, so $-1 < 3x - 1 < 2$. Thus f gives a function from $(0,1)$ to $(-1,2)$. We show f is 1-1. Suppose $x_1, x_2 \in (0,1)$ with $f(x_1) = f(x_2)$. Then $3x_1 - 1 = 3x_2 - 1$. Add 1 to both sides and divide by 3 to see that $x_1 = x_2$. Thus f is 1-1. We show f is onto. Take $y \in (-1, 2)$. Observe that $0 < y + 1 < 3$ so $0 < \frac{y+1}{3} < 1$ and $f\left(\frac{y+1}{3}\right) = 3\left(\frac{y+1}{3}\right) - 1 = y$. We conclude that f is onto.

(5) Proof by contradiction. Suppose $(0,1)$ is countable. Indeed suppose the elements of $(0,1)$ are $a_1, a_2, a_3, \ldots$ Expand these numbers in their decimal expansions:

$a_1 = .a_{11}\, a_{12}\, a_{13} \cdots$

$a_2 = .a_{21}\, a_{22}\, a_{23} \cdots$

$a_3 = .a_{31}\, a_{32}\, a_{33} \cdots$

$\vdots$

I will exhibit a number $\in (0,1)$ which is not in the list of all numbers from $(0,1)$.

Let $b = .b_1\, b_2\, b_3 \ldots$ where $b_1 \neq a_{11}, 0$ or 9, $b_2 \neq a_{22}, 0$ or 9, etc. in general,