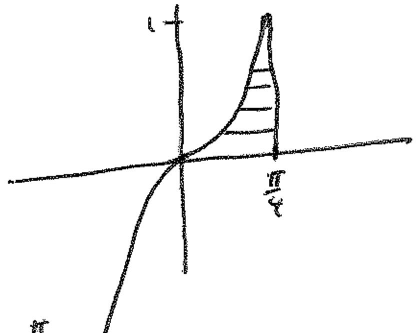
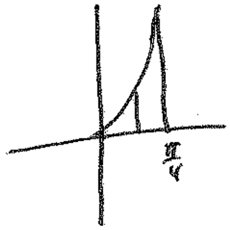
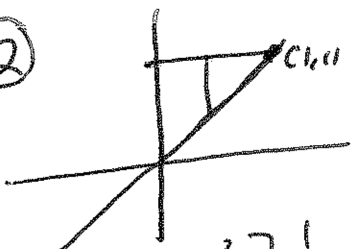
① $x=\tan^{-1}y$ means $\tan x = y$ $\quad \tan\frac{\pi}{4}=1$ 2000 – final





$$\int_0^{\frac{\pi}{4}}\int_0^{\tan x} \sec^5 x \, dy\, dx = \int_0^{\frac{\pi}{4}} \sec^4 x \sec x \tan x \, dx = \left.\frac{\sec^5 x}{5}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{5}\left[\frac{1}{(\frac{\sqrt{2}}{2})^5} - 1\right] = \frac{4\sqrt{2}-1}{5}$$

② 

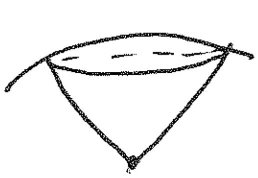
$$\int_0^1\int_0^y e^{-y^2}\,dx\,dy = \int_0^1 y e^{-y^2}\,dy$$

$$\left.-\frac{1}{2}e^{-y^2}\right]_0^1 = \frac{1}{2}\left[1-\frac{1}{e}\right]$$

③ $u=\frac{x}{a}$ $\quad v=\frac{y}{b}$

$$\iint_{u^2+v^2=1} u^2+v^2 \left|\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}\right| \underbrace{du\,dv} = ab \iint_{u^2+v^2\le 1} u^2+v^2 \underset{iso}{du\,dv}$$

$$= ab\int_0^1\int_0^{2\pi} r^3 \, d\theta\, dr = 2\pi ab \left.\frac{r^4}{4}\right]_0^1 = \frac{\pi ab}{2}$$

④ 

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^1 \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta = \frac{2\pi}{3}\left.(-\cos\varphi)\right]_0^{\frac{\pi}{4}} = \frac{2\pi}{3}\left(1-\frac{\sqrt{2}}{2}\right)$$

$z=r$ is $\varphi=\frac{\pi}{4}$ 

⑤ $u=x+y$ $\quad v=x-y$

$$\int_{-1}^{2}\int_1^{5} u^3 e^v \left|\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}\right| du\,dv = \frac{1}{2}\left.\frac{u^4}{4}\right]_1^5 \left.e^v\right]_{-1}^2$$

$$= \frac{1}{8}(625-1)\left(e^2-\frac{1}{e}\right)$$

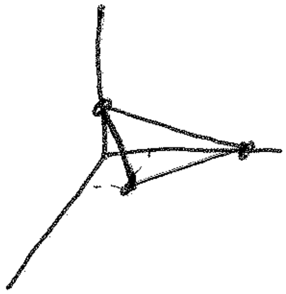
$\frac{u+v}{2}=x$

$\frac{u-v}{2}=y$

$$\left|\begin{vmatrix}\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2}\end{vmatrix}\right| = \left|-\frac{1}{4}-\frac{1}{4}\right| = \frac{1}{2}$$

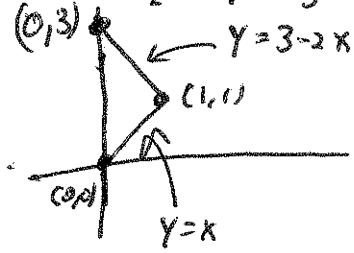
⑥ $\int_0^{2\pi}\int_0^1 r e^{r^2}\,dr\,d\theta = 2\pi \tfrac{1}{2} e^{r^2}\Big]_0^1 = \pi(e-1)$

⑦ The plane which contains $(1,1,0)$, $(0,3,0)$ and $(0,0,1)$ is $2x+y+3z=3$, because all 3 points satisfy this equation.



I parametrize the triangle by

$$x = x$$
$$y = y$$
$$z = 1 - \tfrac{2}{3}x - \tfrac{1}{3}y$$



$$\text{integral} = \int_0^1 \int_x^{3-2x} x \sqrt{\underbrace{\begin{vmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \end{vmatrix}^2}_{1} + \underbrace{\begin{vmatrix} 1 & 0 \\ -\frac{2}{3} & -\frac{1}{3} \end{vmatrix}^2}_{\frac{1}{9}} + \underbrace{\begin{vmatrix} 0 & 1 \\ -\frac{2}{3} & -\frac{1}{3} \end{vmatrix}^2}_{\frac{4}{9}}}\; dy\,dx$$

$$= \frac{\sqrt{14}}{3}\int_0^1 -x^2 + 3x - 2x^2\,dx = \sqrt{14}\int_0^1 x - x^2\,dx = \sqrt{14}\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big]_0^1$$

$$= \sqrt{14}\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\sqrt{14}}{6}$$

⑧ $\vec{F} = \vec{\nabla}\left(e^x \sin y + \frac{z^3}{3}\right)$ so $\text{int} = e^x \sin y + \frac{z^3}{3}\Big]_{(0,0,1)}^{(1,1,e)}$

$$= e\sin 1 + \frac{e^3}{3} - \frac{1}{3}$$

⑨ $\underset{\text{Stokes}}{=} \int_{x^2+y^2=1} \vec{F}\cdot d\vec{r} = \int_0^{2\pi}\left(y\frac{dx}{dt} - x\frac{dy}{dt}\right)dt = \int_0^{2\pi} -\sin^2 t - \cos^2 t\,dt = -2\pi$

$x = \cos t$
$y = \sin t$

⑩

$\underset{\text{Green}}{=} \iint_{\text{inside}} \frac{\partial}{\partial x}(8x - 15y) - \frac{\partial}{\partial y}(3y + x)\,\underbrace{dx\,dy}_{\text{iso}} = 5\iint_{\text{inside}} dx\,dy = 5\,\text{area}$

$$= 5\left[\tfrac{1}{2}\pi(2)^2 - \tfrac{1}{2}\pi(1)^2\right] = \frac{5\pi}{2}(3) = \frac{15\pi}{2}$$