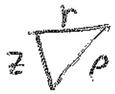
2000 Exam 4

(1) $x = 5\sin\varphi\cos\theta$
$y = 2 + 5\sin\varphi\sin\theta$
$z = 5\cos\varphi$



(2) gradients are $\perp$ to level sets. My surface is level zero of $x^2+y^2-z=0$. gradient L its $\big|_{(1,2,5)} = 2x\vec{i} + 2y\vec{j} - \vec{k}\,\big|_{(1,2,5)}$
$= 2\vec{i} + 4\vec{j} - \vec{k}$. The tang. plane is $2(x-1)+4(y-2)-(z-5)=0$.

(3) Parametrize using $x=x$, $y=y$, $z=x^2+y^2$ for (x,y) inside $x^2+y^2=1$.

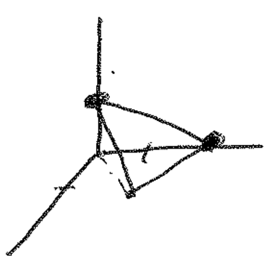
$$\text{Surface area} = \iint_{\text{inside circle}} \sqrt{\begin{vmatrix}1&0\\0&1\end{vmatrix}^2 + \begin{vmatrix}1&0\\2x&2y\end{vmatrix}^2 + \begin{vmatrix}0&1\\2x&2y\end{vmatrix}^2}\;\underbrace{dx\,dy}_{\text{iso}}$$

$$= \iint_{\text{inside circle}} \sqrt{1+4y^2+4x^2}\;dx\,dy = \int_0^1\int_0^{2\pi} r\sqrt{1+4r^2}\,d\theta\,dr$$

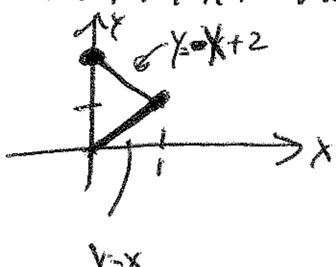
change to polar coordinates

$$= 2\pi\,\frac{1}{8}\,\frac{2}{3}(1+4r^2)^{\frac{3}{2}}\Big]_0^1 = \frac{\pi}{6}\left((5)^{\frac{3}{2}}-1\right)$$

(4)



The plane is $x+y+2z=2$.
i.e. $z = 1 - \frac{x}{2} - \frac{y}{2}$
The parameter space is



$$= \int_0^1\int_x^{2-x} x\sqrt{\begin{vmatrix}1&0\\-\frac{1}{2}&-\frac{1}{2}\end{vmatrix}^2 + \begin{vmatrix}1&0\\0&1\end{vmatrix}^2 + \begin{vmatrix}0&1\\-\frac{1}{2}&-\frac{1}{2}\end{vmatrix}^2}\;dy\,dx$$

$$= \int_0^1 xy\Big]_x^{2-x} dx\,\sqrt{1+\frac{2}{4}} = \frac{\sqrt{6}}{2}\int_0^1 x(2-x-x)\,dx = \sqrt{6}\int_0^1 x(1-x)\,dx = \sqrt{6}\int_0^1 x-x^2\,dx$$

$$= \sqrt{6}\left(\frac{x^2}{2}-\frac{x^3}{3}\right)\Big]_0^1 = \sqrt{6}\left(\frac{1}{2}-\frac{1}{3}\right) = \boxed{\frac{\sqrt{6}}{6}}$$

(5) $T_\varphi \times T_\theta = \sin\varphi \, (x\vec{i} + y\vec{j} + z\vec{k})$

$$\iint_S \vec{F} \cdot d\vec{S} = \int_0^{\pi}\int_0^{2\pi} \sin\varphi \, (-2)(x^2+y^2+z^2) \, d\theta \, d\varphi = \int_0^{\pi} (-4) \sin\varphi \, d\varphi$$

$$= +4 \cos\varphi \Big]_0^{\pi} = -8\pi$$