1999-4

Page 1

① Is $\vec{c}\,'(t) = \vec{F}(\vec{c}(t))$?

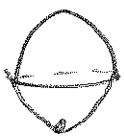
LHS $= (2\cos 2t, -2\sin 2t)$ ← Not equal

RHS $= \vec{F}(\sin 2t, \cos 2t) = (\cos 2t, -\sin 2t)$

$\therefore$ $\vec{c}$ is <u>not</u> a flow line of $\vec{F}$.

② $\vec{c}(2) = (9, 16, 10)$ $\quad \vec{c}\,'(t) = (3t^2, 8t, 5)$ $\quad \vec{c}\,'(2) = (12, 16, 5)$

The tan line is $\dfrac{x-9}{12} = \dfrac{y-16}{16} = \dfrac{(z-10)}{5}$

④ 

$z = 32 - r^2$

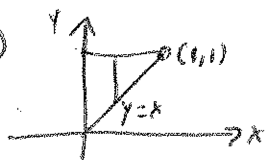
$z = r^2$

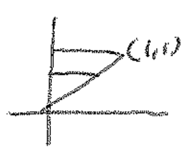
← The intersection is
$32 - r^2 = r^2$
$16 = r^2$
$4 = r$
$16 = z$

$$Vol = \int_0^{2\pi}\int_0^{4}\int_{r^2}^{32-r^2} r\,dz\,dr\,d\theta$$

$$= 2\pi \int_0^4 32r - 2r^3\,dr = 2\pi\left(16r^2 - \frac{r^4}{2}\right)\Big|_0^4 = 2\pi\,\frac{4^4}{2} = 256\pi$$

③ $$\int_{2\pi}^{4\pi} \|c'(t)\|\,dt = \int_{2\pi}^{4\pi} \|(0,1,1)\|\,dt = \sqrt{2}\,(2\pi)$$

⑤ 



$$\int_0^1\int_0^y e^{y^2}\,dx\,dy = \int_0^1 y e^{y^2}\,dy = \frac{1}{2}e^{y^2}\Big]_0^1$$

$$= \frac{1}{2}(e-1)$$

⑥ $$\overrightarrow{(3,0,5)(1,1,1)} \times \overrightarrow{(3,0,5)(2,3,4)} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 1 & -4 \\ -1 & 3 & -1 \end{vmatrix} = \vec{i}(11) - \vec{j}(-2) + \vec{k}(-5)$$

$11(x-1) + 2(y-1) - 5(z-1) = 0$ $\qquad$ $11x + 2y - 5z = 8$

⑦ $\overrightarrow{(1,2,4)(7,8,9)} = 6\vec{i} + 6\vec{j} + 5\vec{k}$ $\qquad$ $\dfrac{x-1}{6} = \dfrac{y-2}{6} = \dfrac{z-4}{5}$

⑧ $x = 3t - 2 \quad y = 4t + 3 \quad z = t - 1$

$(3t-2) - 2(4t+3) + 3(t-1) + 7 = 0$

$(3 - 8 + 3)t - 2 - 6 - 3 + 7 = 0$

$-2t - 4 = 0$

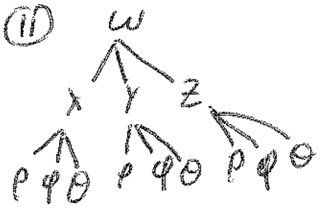
$-2 = t$

$x = -8$
$y = -5$
$z = -3$

(9) The point is $(1,2,5)$. $\vec\nabla$'s are $\perp$ level sets. My surface is
$0 = x^2+y^2-z$. $\vec\nabla(\text{RHS})\big|_{(1,2,5)} = 2x\vec i + 2y\vec j - \vec k\big|_{(1,2,5)} = 2\vec i + 4\vec j - \vec k$

$2(x-1)+4(y-2)-(z-5)=0$

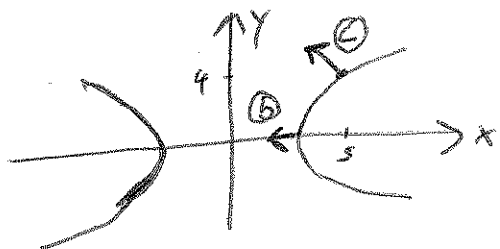
(10) $\|\vec c(t)\| = \text{constant}$ so $\vec c'\cdot\vec c' = \text{constant}$ so $\frac{d}{dt}(\vec c'\cdot\vec c')=0$
$\therefore 2\vec c''\cdot\vec c' = 0$ $\therefore \vec c''\cdot\vec c' = 0$ Thus $\vec c'' \perp \vec c'$.

(11) 

$x = \rho\cos\theta\sin\varphi$
$y = \rho\sin\theta\sin\varphi$
$z = \rho\cos\varphi$

$$\frac{\partial w}{\partial\theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial\theta} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial\theta}$$

$$\frac{\partial w}{\partial\theta} = \frac{\partial w}{\partial x}(-\rho\sin\theta\sin\varphi) + \frac{\partial w}{\partial y}(\rho\cos\theta\sin\varphi)$$

(12) 

$9 = x^2 - y^2$

ⓑ $\vec\nabla f = 2x\vec i - 2y\vec j$ $\vec\nabla f\big|_{(3,0)} = 6\vec i$ $-\frac{1}{10}\vec\nabla f\big|_{(3,0)} = -.6\vec i$

ⓒ $\vec\nabla f\big|_{(5,4)} = 10\vec i - 8\vec j$ $-\frac{1}{10}\vec\nabla f\big|_{(5,4)} = -\vec i + .8\vec j$

(13) The point is $(1,0,1)$
The normal vector is $\vec T_u\times\vec T_v\big|_{(1,0)} = (2u\cos v, 2u\sin v, 1)\times(-u^2\sin v, u^2\cos v, 0)\big|_{(1,0)}$

$$= (2,0,1)\times(0,1,0) = \begin{vmatrix}\vec i & \vec j & \vec k\\ 2 & 0 & 1\\ 0 & 1 & 0\end{vmatrix} = -\vec i + 2\vec k$$

$-(x-1)+2(z-1)=0$ $\qquad -x+2z+1-2=0$
$-x+2z=1$

(14) The answer is $2\cdot3\cdot\pi$ use Green's Theorem or Change of variables thm.
Here is Green's thm:

$$\iint_{\text{inside ellipse}} 1\,dx\,dy = \frac12\int_{\text{bdry}} -y\,dx + x\,dy = \frac12\int_0^{2\pi}\big((-3\sin\theta)(-2\sin\theta) + 2\cos\theta\, 3\cos\theta\big)\,d\theta = \frac12\cdot6\int_0^{2\pi}d\theta$$

$= 6\pi$

$x = 2\cos\theta$
$y = 3\sin\theta$

15. Do it directly or use Stoke's Theorem:

$$\iint_S (\vec\nabla \times \vec F)\cdot d\vec S = \int_{\text{bdry}} \vec F \cdot d\vec\alpha = \int_0^{2\pi}\left(y\frac{dx}{d\theta} - x\frac{dy}{d\theta}\right)d\theta = \int_0^{2\pi}(-\sin\theta\sin\theta - \cos\theta\cos\theta)\,d\theta$$

$x=\cos\theta$
$y=\sin\theta$
$z=0$

$= -2\pi$

$\boxed{\text{Page 3}}$

16. $$\int (3y+x)\,dx + (8x-15y)\,dy = \iint_{\text{inside}} (8-3)\,dx\,dy = 5 \text{ area inside } C = 5\,\frac{\pi}{2}(2^2-1^2) = \frac{15\pi}{2}$$

Green's Thm

17. Do it directly or use Stoke's Thm

$$\int_C \vec F \cdot d\vec s = \iint_{\substack{\text{inside}\\\text{ellipse}}} \vec\nabla\times\vec F\cdot d\vec S = \iint_{\substack{\text{inside}\\\text{ellipse}}} \begin{vmatrix} \vec i & \vec j & \vec k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & x & 3y \end{vmatrix} \cdot d\vec S = \iint_{\substack{\text{inside}\\\text{ellipse}}} (3\vec i + 2\vec j + \vec k)\cdot(\vec T_u \times \vec T_v)\,du\,dv$$

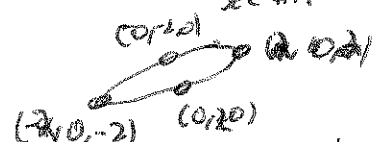
where $\Phi(u,v)$ is yet to be determined

$$= \iint_{\substack{\text{inside}\\\text{ellipse}}} (3\vec i + 2\vec j + \vec k)\cdot \frac{\vec T_u \times \vec T_v}{\|\vec T_u \times \vec T_v\|}\,\|\vec T_u \times \vec T_v\|\,du\,dv = \iint_{\substack{\text{inside}\\\text{ellipse}}} (3\vec i + 2\vec j + \vec k)\cdot\frac{(\vec i - \vec k)}{\sqrt 2}\,\|\vec T_u \times \vec T_v\|\,du\,dv$$

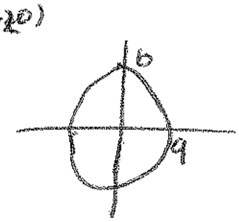
This is a unit vector $\perp$ to the ellipse but the ellipse lives in the plane $z=x$ so this is $\frac{\vec i - \vec k}{\sqrt 2}$

$$= \frac{2}{\sqrt 2}\iint_{\substack{\text{inside}\\\text{ellipse}}} \|\vec T_u \times \vec T_v\|\,du\,dv = \frac{2}{\sqrt 2}\cdot\text{area of ellipse} = \frac{2}{\sqrt 2}\,\pi(2)\,2\sqrt 2 = 8\pi$$

see #14

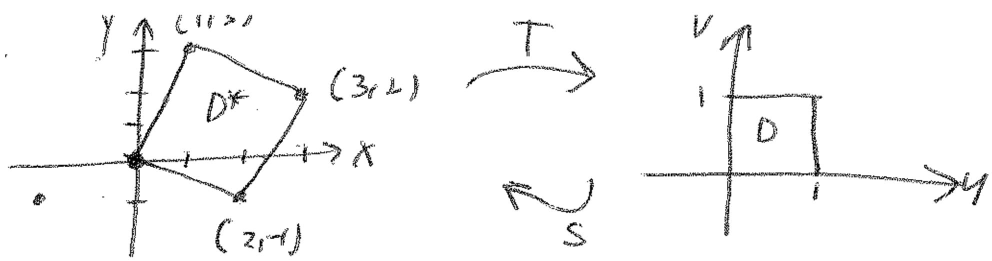


the area inside



is $ab\pi$
here $a=2$ $b=2\sqrt 2$

(18)



Page 4

A good S is $\begin{bmatrix}u\\v\end{bmatrix} \longmapsto \begin{bmatrix}2&1\\-1&3\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}$

A good T is $\begin{bmatrix}x\\y\end{bmatrix} \longmapsto \frac{1}{7}\begin{bmatrix}3&-1\\1&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$

That is $T(x,y) = \left(\frac{3}{7}x - \frac{1}{7}y,\ \frac{1}{7}x + \frac{2}{7}y\right)$

check $T(0,0) = (0,0)$

$T(2,-1) = \left(\frac{6}{7} + \frac{1}{7},\ \frac{2}{7} - \frac{2}{7}\right) = (1,0)$

$T(3,2) = \left(\frac{9}{7} - \frac{2}{7},\ \frac{3}{7} + \frac{4}{7}\right) = (1,1)$

$T(1,3) = \left(\frac{3}{7} - \frac{3}{7},\ \frac{1}{7} + \frac{6}{7}\right) = (0,1)$ ✓

(19) $\iint_D (x-y)\,dx\,dy = \int_1^2 \int_0^1 [u - (v + uv)] \begin{vmatrix} 1 & \\ & 1+u \end{vmatrix} du\,dv$

$= \int_1^2 \int_0^1 u(1+u) - v(1+u)^2\,du\,dv$

$= \int_1^2 \frac{u^2}{2} + \frac{u^3}{3} - v\frac{(1+u)^3}{3}\Big]_0^1\,dv$

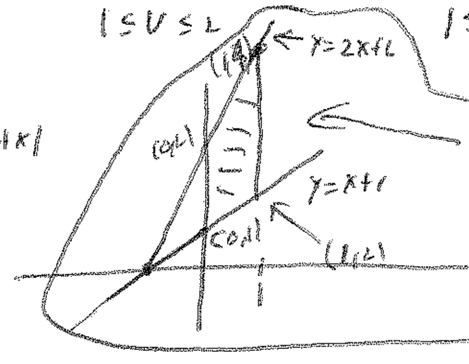
5

$= \int_1^2 \frac{1}{2} + \frac{1}{3} - v\left(\frac{8}{3} - \frac{1}{3}\right)\,dv$

$= \frac{5}{6}v - \frac{v^2}{2}\left(\frac{7}{3}\right)\Big]_1^2 = \frac{5}{6} - \frac{7}{3}\left(2 - \frac{1}{2}\right) = \frac{5}{6} - \frac{7}{3}\left(\frac{3}{2}\right) = -\frac{16}{6}$

$x = u$, $y = v(1+u)$ so $u = x$, $v = \frac{y}{1+x}$ so $0 \le u \le 1$, $1 \le v \le 2$ becomes $0 \le x \le 1$, $1 \le \frac{y}{1+x} \le 2$

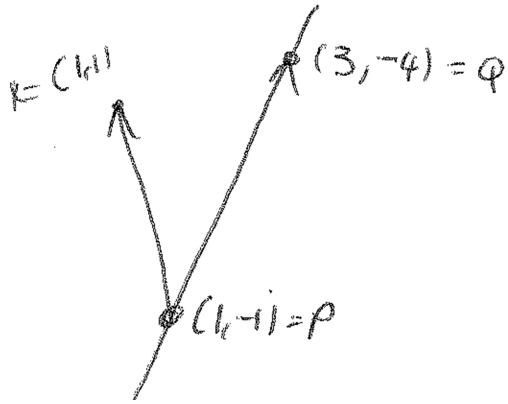
so $D = T(D^*)$ is $0 \le x \le 1$, $1 + x \le y \le 2(1+x)$



this is D

5

(20)



$\overrightarrow{PR} = 2\vec{j}$

$\overrightarrow{PQ} = 2\vec{i} - 3\vec{j}$

$\boxed{\text{Page 5}}$

$$\text{dist} = \left\| \overrightarrow{PR} - \text{proj}_{\overrightarrow{PQ}} \overrightarrow{PR} \right\| = \left\| 2\vec{j} - \frac{-6}{13}(2\vec{i} - 3\vec{j}) \right\| = \left\| \frac{-12}{13}\vec{i} + \frac{8}{13}\vec{j} \right\|$$

$$= \frac{\sqrt{208}}{13} = \frac{4\sqrt{13}}{13} = \frac{4}{\sqrt{13}}$$