Math 550, Exam 1, solution, Spring 2013

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: turn the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. SHOW your work. $\boxed{\textit{CIRCLE}}$ your answer. **CHECK** your answer whenever possible.

No Calculators or Cell phones.

The solutions will be posted later today.

1. (9 points) **Compute** $\int_0^1 \int_y^1 \sin(x^2) dx dy$ **. Explain very carefully what you are doing.**

None of us know an elementary anti-derivative for $\sin(x^2)$. Lets see if things get better after we exchange the order of integration. A picture is available on a different page. The original integral is equal to

$$\int_0^1 \int_0^x \sin(x^2) dy dx = \int_0^1 \sin(x^2) y \Big|_0^x dx = \int_0^1 \sin(x^2) x dx = \frac{-\cos(x^2)}{2}\Big|_0^1$$

$$= \boxed{\frac{1}{2}(1 - \cos(1))}.$$

2. (9 points) **Let** $f(x)$ **be a continuous function for** $a \le x \le b$ **. Find a formula which relates** $(\int_a^b f(x)dx)^2$ **and** $\int_a^b \int_x^b f(x)f(y)dydx$ **. Explain why your formula is correct very carefully.**

We see that

$$(\textstyle\int_a^b f(x)dx)^2 =_1 (\int_a^b f(x)dx)(\int_a^b f(x)dx) =_2 (\int_a^b f(x)dx)(\int_a^b f(y)dy)$$

$$=_3 (\textstyle\int_a^b f(x)(\int_a^b f(y)dy)dx) =_4 (\int_a^b(\int_a^b f(x)f(y)dy)dx)$$

$$=_5 \textstyle\int\int_{[a,b]\times[a,b]} f(x)f(y)dA.$$

The equalities 1 and 2 are obvious. For equality 3, it is legal to move the constant $\int_a^b f(y)dy$ inside the integral $\int_a^b f(x)dx$. For equality 4, as far as the integral $\int_a^b f(y)dy$ is concerned, $f(x)$ is a constant. It is legal to move the constant inside the integral sign. The left side of equality 4 is an iterated integral; the right side is the corresponding double integral. We split the rectangle $[a,b]\times[a,b]$ into two triangles by drawing the line connecting the corner (a,a) to the corner (b,b). (I put a picture on a different page.)

$$=_6 \begin{cases} \int\int_{\text{the triangle with vertices (a,a),(a,b),(b,b)}} f(x)f(y)dA \\ +\int\int_{\text{the triangle with vertices (a,a),(b,a),(b,b)}} f(x)f(y)dA \end{cases}$$

We fill up the triangle of the first integral using vertical lines. We fill up the triangle of the second integral using horizontal lines.

$$=_7 \int_a^b \int_x^b f(x)f(y)dydx + \int_a^b \int_y^b f(x)f(y)dxdy$$

$$=_8 \int_a^b \int_x^b f(x)f(y)dydx + \int_a^b \int_x^b f(y)f(x)dydx = 2\int_a^b \int_x^b f(x)f(y)dydx.$$

In 8, we replaced all the x's by y's and all of the y's by x's in the second integral.

We have shown that

$$\boxed{(\int_a^b f(x)dx)^2 = 2\int_a^b \int_x^b f(x)f(y)dydx}$$

3. (8 points) **A lumberjack cuts a wedge-shaped piece W out of a cylindrical tree of radius a by making two saw cuts. The first cut is parallel to the ground. The second cut makes an angle θ with the first cut and meets the first cut along a diagonal of the circle that contains the first cut. Find the volume of W. Explain very carefully what you are doing.**

We use a triple integral. The outer two integrals are over the base. The inner integral is from the bottom ($z=0$) to the top ($z = x\tan\theta$). The base is the semi-circle with positive x and inside $x^2+y^2=a^2$. I drew a picture elsewhere. The volume of W is

$$\int_{-a}^{a}\int_0^{\sqrt{a^2-y^2}}\int_0^{x\tan\theta} dzdxdy = \int_{-a}^{a}\int_0^{\sqrt{a^2-y^2}} z\Big|_0^{x\tan\theta} dxdy$$

$$= \int_{-a}^{a}\int_0^{\sqrt{a^2-y^2}} x\tan\theta dxdy = \int_{-a}^{a} \frac{x^2}{2}\tan\theta\Big|_0^{\sqrt{a^2-y^2}} dy = \int_{-a}^{a} \frac{a^2-y^2}{2}\tan\theta dy$$

$$= \left(\frac{a^2y}{2} - \frac{y^3}{6}\right)\tan\theta\Big|_{-a}^{a} = 2\left(\frac{a^3}{2} - \frac{a^3}{6}\right)\tan\theta = \boxed{\frac{2a^3\tan\theta}{3}}$$

4. (8 points) **Let $f(x,y,z)$ be a continuous function which is defined on all of three space. Let a, b, and c be constants. Consider the function $F(x)=\int_c^x\int_a^b f(x,y,z)dydz$. Find an expression for $\frac{d}{dx}F(x)$ in which all differentiation is done before all integration. Explain very carefully what you are doing.**

We use the chain rule. View F as a function of u and v, where $u(x)=x$ and $v(x)=x$ and $F(u,v)=\int_c^u\int_a^b f(v,y,z)dydz$. The chain rule is $\frac{d}{dx}F(x)=\frac{\partial F}{\partial u}\frac{du}{dx}+\frac{\partial F}{\partial v}\frac{dv}{dx}$. It is clear that $\frac{du}{dx}=\frac{dv}{dx}=1$. To compute $\frac{\partial F}{\partial u}$ we use the Fundamental Theorem of Calculus which says that $\frac{d}{du}\int_c^u g(z)dz=g(u)$. For us, $g(z)$ is the function $\int_a^b f(v,y,z)dy$, where v is a constant as far is the calculation $\frac{\partial F}{\partial u}$ is concerned. So $\frac{\partial F}{\partial u}=\int_a^b f(v,y,u)dy$. To compute $\frac{\partial F}{\partial v}$ we differentiate under the integral sign twice:

$$\frac{\partial F}{\partial v}=\frac{\partial}{\partial v}\int_c^u\int_a^b f(v,y,z)dydz=\int_c^u\frac{\partial}{\partial v}\int_a^b f(v,y,z)dydz$$

$$=\int_c^u\int_a^b\frac{\partial}{\partial v}f(v,y,z)dydz=\int_c^u\int_a^b f_v(v,y,z)dydz.$$

We have shown that

$$\frac{d}{dx}F(x)=\frac{\partial F}{\partial u}\frac{du}{dx}+\frac{\partial F}{\partial v}\frac{dv}{dx}$$

$$=\int_a^b f(v,y,u)dy+\int_c^u\int_a^b f_v(v,y,z)dydz=\int_a^b f(x,y,x)dy+\int_c^x\int_a^b f_x(x,y,z)dydz.$$

We conclude

$$\boxed{\frac{d}{dx}F(x)=\int_a^b f(x,y,x)dy+\int_c^x\int_a^b f_x(x,y,z)dydz.}$$

5. (8 points) **Find a linear map $L\colon\mathbb{R}^2\to\mathbb{R}^2$ which carries the parallelogram with vertices $(0,0)$, (a,b), (c,d), $(a+c,b+d)$ to the parallelogram with vertices $(0,0)$, (e,f), (g,h), $(e+g,f+h)$. (You may assume that both parallelograms are honest-to-goodness parallelograms.) Explain very carefully what you are doing.**

We take L to be the transformation $L=S\circ T^{-1}$ where T is the transformation that carries the unit square to the parallelogram with vertices $(0,0)$, (a,b), (c,d), $(a+c,b+d)$ and S is the transformation that carries the unit square to parallelogram with vertices $(0,0)$, (e,f), (g,h), $(e+g,f+h)$. Thus

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}a&c\\b&d\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}\quad S\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}e&g\\f&h\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix},$$

and

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{1}{ad-bc}\begin{bmatrix} e & g \\ f & h \end{bmatrix}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ed-gb & ga-ec \\ fd-hb & ha-fc \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

We conclude that

$$\boxed{L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{1}{ad-bc}\begin{bmatrix} ed-gb & ga-ec \\ fd-hb & ha-fc \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}}.$$

6. (8 points) **What is the area of the parallelogram with vertices** $(0,0)$**,** (a,b)**,** (c,d)**,** $(a+c,b+d)$**? (You may assume that the parallelogram is an honest-to-goodness parallelogram.) Explain very carefully what you are doing.**

We calculated in class that the area is $|\det\begin{bmatrix} a & c \\ b & d \end{bmatrix}| = \boxed{|ad-bc|}$. Our argument went something like this. Let v be the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ and w be the vector $\begin{bmatrix} c \\ d \end{bmatrix}$. The area of the parallelogram determined by v and w is the length of the base time the height. We take v to be the base. Then the height is the length of w minus the projection of w onto v. (I have drawn a picture.) The area is

$$||v||\,||(w-\operatorname{proj}_v w)|| = ||v||\,||(w-\frac{v\cdot w}{v\cdot v}v)|| = \sqrt{(v\cdot v)(w-\frac{v\cdot w}{v\cdot v}v)\cdot(w-\frac{v\cdot w}{v\cdot v}v)}$$

$$= \sqrt{(v\cdot v)(w\cdot w-2\frac{v\cdot w}{v\cdot v}v\cdot w+\left(\frac{v\cdot w}{v\cdot v}\right)^2 v\cdot v)}$$

$$= \sqrt{(v\cdot v)(w\cdot w)-2(v\cdot w)^2+(v\cdot w)^2}$$

$$= \sqrt{(v\cdot v)(w\cdot w)-(v\cdot w)^2}$$

$$= \sqrt{(a^2+b^2)(c^2+d^2)-(ac+bd)^2}$$

$$= \sqrt{(a^2c^2+a^2d^2+b^2c^2+b^2d^2)-(a^2c^2+2abcd+b^2d^2)}$$

$$= \sqrt{a^2d^2+b^2c^2-2abcd}$$

$$= \sqrt{(ad-bc)^2} = |ad-bc|$$