exercises

1. Evaluate $\omega \wedge \eta$ if

(a) $\omega = 2x\,dx + y\,dy$
$\eta = x^3\,dx + y^2 dy$

(b) $\omega = x\,dx - y\,dy$
$\eta = y\,dx + x\,dy$

(c) $\omega = x\,dx + y\,dy + z\,dz$
$\eta = z\,dx\,dy + x\,dy\,dz + y\,dz\,dx$

(d) $\omega = xy\,dy\,dz + x^2\,dx\,dy$
$\eta = dx + dz$

(e) $\omega = e^{xyz}\,dx\,dy$
$\eta = e^{-xyz}\,dz$

2. Prove that

$$(a_1\,dx + a_2\,dy + a_3\,dz) \wedge (b_1\,dy\,dz + b_2\,dz\,dx + b_3\,dx\,dy) = \left(\sum_{i=1}^{3} a_i b_i\right) dx\,dy\,dz.$$

3. Find $d\omega$ in the following examples:

(a) $\omega = x^2 y + y^3$
(b) $\omega = y^2 \cos x\,dy + xy\,dx + dz$
(c) $\omega = xy\,dy + (x+y)^2\,dx$
(d) $\omega = x\,dx\,dy + z\,dy\,dz + y\,dz\,dx$
(e) $\omega = (x^2 + y^2)\,dy\,dz$
(f) $\omega = (x^2 + y^2 + z^2)\,dz$
(g) $\omega = \dfrac{-x}{x^2+y^2}\,dx + \dfrac{y}{x^2+y^2}\,dy$
(h) $\omega = x^2 y\,dy\,dz$

4. Let C be the line segment from the point $(-2, 0, 1)$ to $(3, 6, 9)$. Let $\omega_1 = y\,dx + x\,dy + xy\,dz$, $\omega_2 = z\,dx + y\,dy + 2x\,dz$, and $f(x, y, z) = xy$. Calculate the following:

(a) $\displaystyle\int_C f\omega_1$ (b) $\displaystyle\int_C f\omega_2$ (c) $\displaystyle\int_C \omega_1 + \omega_2$

5. Let C be parameterized by $c(t) = (t^2 + 4t,\ t + 1)$, $t \in [0, \pi]$. Let $\omega_1 = y\,dx + x\,dy$, $\omega_2 = y^2 dx + x^2 dy$, and $f(x, y) = x$. Calculate the following:

(a) $\displaystyle\int_C f\omega_1$ (b) $\displaystyle\int_C f\omega_2$ (c) $\displaystyle\int_C \omega_1 + \omega_2$

6. Let $\mathbf{V}\colon K \to \mathbb{R}^3$ be a vector field defined by $\mathbf{V}(x, y, z) = G(x, y, z)\mathbf{i} + H(x, y, z)\mathbf{j} + F(x, y, z)\mathbf{k}$, and let η be the 2-form on K given by

$$\eta = F\,dx\,dy + G\,dy\,dz + H\,dz\,dx.$$

Show that $d\eta = (\operatorname{div} \mathbf{V})\,dx\,dy\,dz$.

7. If $\mathbf{V} = A(x, y, z)\mathbf{i} + B(x, y, z)\mathbf{j} + C(x, y, z)\mathbf{k}$ is a vector field on $K \subset \mathbb{R}^3$, define the operation Form$_2$: Vector Fields $\to$ 2-forms by

$$\text{Form}_2(\mathbf{V}) = A\,dy\,dz + B\,dz\,dx + C\,dx\,dy.$$

(a) Show that
$\text{Form}_2(\alpha\mathbf{V}_1 + \mathbf{V}_2) = \alpha\,\text{Form}_2(\mathbf{V}_1) + \text{Form}_2(\mathbf{V}_2)$, where α is a real number.

(b) Show that $\text{Form}_2(\text{curl}\,\mathbf{V}) = d\omega$, where $\omega = A\,dx + B\,dy + C\,dz$.

8. Using the differential form version of Stokes' theorem, prove the vector field version in Section 8.2. Repeat for Gauss' theorem.

9. Interpret Theorem 16 in the case $k = 1$.

10. Let $\omega = (x + y)\,dz + (y + z)\,dx + (x + z)\,dy$, and let S be the upper part of the unit sphere; that is, S is the set of (x, y, z) with $x^2 + y^2 + z^2 = 1$ and $z \ge 0$. ∂S is the unit circle in the xy plane. Evaluate $\int_{\partial S} \omega$ both directly and by Stokes' theorem.

11. Let T be the triangular solid bounded by the xy plane, the xz plane, the yz plane, and the plane $2x + 3y + 6z = 12$. Compute

$$\iint_{\partial T} F_1\,dx\,dy + F_2\,dy\,dz + F_3\,dz\,dx$$

directly and by Gauss' theorem, if

(a) $F_1 = 3y,\ F_2 = 18z,\ F_3 = -12$; and
(b) $F_1 = z,\ F_2 = x^2,\ F_3 = y$.

12. Evaluate $\iint_S \omega$, where $\omega = z\,dx\,dy + x\,dy\,dz + y\,dz\,dx$ and S is the unit sphere, directly and by Gauss' theorem.

13. Let R be an elementary region in $\mathbb{R}^3$. Show that the volume of R is given by the formula

$$v(R) = \frac{1}{3}\iint_{\partial R} x\,dy\,dz + y\,dz\,dx + z\,dx\,dy.$$

14. In Section 4.2, we saw that the length $l(\mathbf{c})$ of a curve $\mathbf{c}(t) = (x(t), y(t), z(t)), a \le t \le b$, was given by the formula

$$l(\mathbf{c}) = \int ds = \int_a^b \left(\frac{ds}{dt}\right) dt$$

where, loosely speaking, $(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$, that is,

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}.$$

Now suppose a surface S is given in parametrized form by $\boldsymbol{\Phi}(u, v) = (x(u, v), y(u, v), z(u, v))$, where $(u, v) \in D$. Show that the area of S can be expressed as

$$A(S) = \iint_D dS,$$

where formally $(dS)^2 = (dx \wedge dy)^2 + (dy \wedge dz)^2 + (dz \wedge dx)^2$, a formula requiring interpretation. [HINT:

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv,$$

and similarly for dy and dz. Use the law of forms for the basic 1-forms du and dv. Then dS turns out to be a function times the basic 2-form $du\,dv$, which we can integrate over D.]

review exercises for chapter 8

1. Let $\mathbf{F} = 2yz\mathbf{i} + (-x + 3y + 2)\mathbf{j} + (x^2 + z)\mathbf{k}$. Evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$, where S is the cylinder $x^2 + y^2 = a^2$, $0 \le z \le 1$ (without the top and bottom). What if the top and bottom are included?

2. Let W be a region in $\mathbb{R}^3$ with boundary ∂W. Prove the identity

$$\iint_{\partial W} [\mathbf{F} \times (\nabla \times \mathbf{G})] \cdot d\mathbf{S} = \iiint_W (\nabla \times \mathbf{F}) \cdot (\nabla \times \mathbf{G})\, dV - \iiint_W \mathbf{F} \cdot (\nabla \times \nabla \times \mathbf{G})\, dV.$$

3. Let $\mathbf{F} = x^2 y\mathbf{i} + z^8\mathbf{j} - 2xyz\mathbf{k}$. Evaluate the integral of $\mathbf{F}$ over the surface of the unit cube.

4. Verify Green's theorem for the line integral

$$\int_C x^2 y\, dx + y\, dy,$$

when C is the boundary of the region between the curves $y = x$ and $y = x^3$, $0 \le x \le 1$.

5. (a) Show that $\mathbf{F} = (x^3 - 2xy^3)\mathbf{i} - 3x^2y^2\mathbf{j}$ is a gradient vector field.

(b) Evaluate the integral of $\mathbf{F}$ along the path $x = \cos^3\theta, y = \sin^3\theta, 0 \le \theta \le \pi/2$.

6. Can you derive Green's theorem in the plane from Gauss' theorem?

7. (a) Show that $\mathbf{F} = 6xy(\cos z)\mathbf{i} + 3x^2(\cos z)\mathbf{j} - 3x^2y(\sin z)\mathbf{k}$ is conservative (see Section 8.3).

(b) Find f such that $\mathbf{F} = \nabla f$.

(c) Evaluate the integral of $\mathbf{F}$ along the curve $x = \cos^3\theta, y = \sin^3\theta, z = 0, 0 \le \theta \le \pi/2$.

8. Let $\mathbf{r}(x, y, z) = (x, y, z), r = \|\mathbf{r}\|$. Show that $\nabla^2(\log r) = 1/r^2$ and $\nabla^2(r^n) = n(n+1)r^{n-2}$.

9. Let the velocity of a fluid be described by $\mathbf{F} = 6xz\mathbf{i} + x^2y\mathbf{j} + yz\mathbf{k}$. Compute the rate at which fluid is leaving the unit cube.

10. Let $\mathbf{F} = x^2\mathbf{i} + (x^2y - 2xy)\mathbf{j} - x^2z\mathbf{k}$. Does there exist a $\mathbf{G}$ such that $\mathbf{F} = \nabla \times \mathbf{G}$?

11. Let $\mathbf{a}$ be a constant vector and $\mathbf{F} = \mathbf{a} \times \mathbf{r}$ [as usual, $\mathbf{r}(x, y, z) = (x, y, z)$]. Is $\mathbf{F}$ conservative? If so, find a potential for it.

12. Show that the fields $\mathbf{F}$ in (a) and (b) are conservative and find a function f such that $\mathbf{F} = \nabla f$.

(a) $\mathbf{F} = \left(y^2 e^{xy^2}\right)\mathbf{i} + \left(2y\, e^{xy^2}\right)\mathbf{j}$

(b) $\mathbf{F} = (\sin y)\mathbf{i} + (x\cos y)\mathbf{i} + (e^z)\mathbf{k}$

13. (a) Let $f(x, y, z) = 3xye^{z^2}$. Compute ∇f.

(b) Let $\mathbf{c}(t) = (3\cos^3 t, \sin^2 t, e^t), 0 \le t \le \pi$. Evaluate

$$\int_{\mathbf{c}} \nabla f \cdot d\mathbf{s}.$$

(c) Verify directly Stokes' theorem for gradient vector fields $\mathbf{F} = \nabla f$.

14. Using Green's theorem, or otherwise, evaluate $\int_C x^3 dy - y^3\, dx$, where C is the unit circle $(x^2 + y^2 = 1)$.

15. Evaluate the integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + 3\mathbf{k}$ and where S is the surface of the unit sphere $x^2 + y^2 + z^2 = 1$.

16. (a) State Stokes' theorem for surfaces in $\mathbb{R}^3$.

(b) Let $\mathbf{F}$ be a vector field on $\mathbb{R}^3$ satisfying $\nabla \times \mathbf{F} = \mathbf{0}$. Use Stokes' theorem to show that $\int_C \mathbf{F} \cdot d\mathbf{s} = 0$ where C is a closed curve.

17. Use Green's theorem to find the area of the loop of the curve $x = a \sin\theta \cos\theta$, $y = a \sin^2\theta$, for $a > 0$ and $0 \le \theta \le \pi$.

18. Evaluate $\int_C yz\, dx + xz\, dy + xy\, dz$, where C is the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the surface $z = y^2$.

19. Evaluate $\int_C (x + y)\, dx + (2x - z)\, dy + (y + z)\, dz$, where C is the perimeter of the triangle connecting $(2, 0, 0)$, $(0, 3, 0)$, and $(0, 0, 6)$, in that order.

20. Which of the following are conservative fields on $\mathbb{R}^3$? For those that are, find a function f such that $\mathbf{F} = \nabla f$.

(a) $\mathbf{F}(x, y, z) = 3x^2 y\mathbf{i} + x^3\mathbf{j} + 5\mathbf{k}$

(b) $\mathbf{F}(x, y, z) = (x + z)\mathbf{i} - (y + z)\mathbf{j} + (x - y)\mathbf{k}$

(c) $\mathbf{F}(x, y, z) = 2xy^3\mathbf{i} + x^2 z^3\mathbf{j} + 3x^2 yz^2\mathbf{k}$

21. Consider the following two vector fields in $\mathbb{R}^3$:

(i) $\mathbf{F}(x, y, z) = y^2\mathbf{i} - z^2\mathbf{j} + x^2\mathbf{k}$

(ii) $\mathbf{G}(x, y, z) = (x^3 - 3xy^2)\mathbf{i} + (y^3 - 3x^2 y)\mathbf{j} + z\mathbf{k}$

(a) Which of these fields (if any) are conservative on $\mathbb{R}^3$? (That is, which are gradient fields?) Give reasons for your answer.

(b) Find potential for the fields that are conservative.

(c) Let α be the path that goes from $(0, 0, 0)$ to $(1, 1, 1)$ by following edges of the cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$ from $(0, 0, 0)$ to $(0, 0, 1)$ to $(0, 1, 1)$ to $(1, 1, 1)$. Let β be the path from $(0, 0, 0)$ to $(1, 1, 1)$ directly along the diagonal of the cube. Find the values of the line integrals

$$\int_\alpha \mathbf{F} \cdot d\mathbf{s}, \quad \int_\alpha \mathbf{G} \cdot d\mathbf{s}, \quad \int_\beta \mathbf{F} \cdot d\mathbf{s}, \quad \int_\beta \mathbf{G} \cdot d\mathbf{s}.$$

22. Consider the *constant* vector field $\mathbf{F}(x, y, z) = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ in $\mathbb{R}^3$.

(a) Find a scalar field $\phi(x, y, z)$ in $\mathbb{R}^3$ such that $\nabla\phi = \mathbf{F}$ in $\mathbb{R}^3$ and $\phi(0, 0, 0) = 0$.

(b) On the sphere Σ of radius 2 about the origin, find all the points at which

(i) ϕ is a maximum, and

(ii) ϕ is a minimum.

(c) Compute the maximum and minimum values of ϕ on Σ.

23. Let $\mathbf{F}$ be a C^1 vector field and suppose $\nabla \cdot \mathbf{F}(x_0, y_0, z_0) > 0$. Show that for a sufficiently small sphere S centered at (x_0, y_0, z_0), the flux of $\mathbf{F}$ out of S is positive.

24. Let $B \subset \mathbb{R}^3$ be a planar region, and let $\mathrm{O} \in \mathbb{R}^3$ be a point. If we connect all points in B to O, we get a cone, say C, with vertex O and base B. Show that

$$\text{Volume } (C) = \frac{1}{3} \text{ area } (B)\, h,$$

where h is the distance of O from the plane of B, using the following steps.

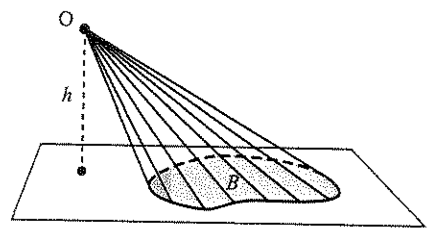

figure 8.R.1

1. Let O be the origin of the coordinate system. Define $\mathbf{r}(x, y, z) := (x, y, z)$. Evaluate the flux of $\mathbf{r}$ through the boundary of C, that is, $\iint_{\partial C} \mathbf{r} \cdot \mathbf{n}\, dA$, where $\mathbf{n}$ is the outward unit normal to ∂C.
2. Evaluate the total divergence $\iiint_C \nabla \cdot \mathbf{r}\, dV$.
3. Use Gauss' theorem, which states that the total divergence of a vector field within a region enclosed by a surface is equal to the flux of that vector field through the surface:

$$\iiint_C \nabla \cdot \mathbf{r}\, dV = \iint_{\partial C} \mathbf{r} \cdot \mathbf{n}\, dA.$$