example 4 Let $\mathbf{c}\colon [1,2] \to \mathbb{R}^2$ be given by $x = e^{t-1}$, $y = \sin(\pi/t)$. Compute the integral

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}} 2x \cos y\, dx - x^2 \sin y\, dy,$$

where $\mathbf{F} = (2x \cos y)\mathbf{i} - (x^2 \sin y)\mathbf{j}$.

solution The endpoints are $\mathbf{c}(1) = (1,0)$ and $\mathbf{c}(2) = (e,1)$. Because $\partial(2x\cos y)/\partial y = \partial(-x^2 \sin y)/\partial x$, $\mathbf{F}$ is irrotational and hence a gradient vector field (as we saw in Example 3). Thus, by Theorem 7, we can replace $\mathbf{c}$ by any piecewise C^1 curve having the same endpoints, in particular, by the polygonal path from $(1,0)$ to $(e,0)$ to $(e,1)$. Thus, the line integral must be equal to

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_1^e 2t \cos 0\, dt + \int_0^1 -e^2 \sin t\, dt = (e^2 - 1) + e^2(\cos 1 - 1)$$
$$= e^2 \cos 1 - 1.$$

Alternatively, using Theorem 3 of Section 7.2, we have

$$\int_{\mathbf{c}} 2x \cos y\, dx - x^2 \sin y\, dy = \int_{\mathbf{c}} \nabla f \cdot d\mathbf{s} = f(\mathbf{c}(2)) - f(\mathbf{c}(1)) = e^2 \cos 1 - 1,$$

because $f(x,y) = x^2 \cos y$ is a potential function for $\mathbf{F}$. Evidently, this technique is simpler than computing the integral directly. ▲

We conclude this section with a theorem that is quite similar in spirit to Theorem 7. Theorem 7 was motivated partly as a converse to the result that curl $\nabla f = \mathbf{0}$ for any C^1 function $f\colon \mathbb{R}^3 \to \mathbb{R}$—or, if curl $\mathbf{F} = 0$, then $\mathbf{F} = \nabla f$. We also know [formula (9) in the table of vector identities in Section 4.4] that div(curl $\mathbf{G}$) $= 0$ for any C^2 vector field $\mathbf{G}$. We can ask about the converse statement: If div $\mathbf{F} = 0$, is $\mathbf{F}$ the curl of a vector field $\mathbf{G}$? The following theorem answers this in the affirmative.

> **Theorem 8** If $\mathbf{F}$ is a C^1 vector field on all of $\mathbb{R}^3$ with div $\mathbf{F} = 0$, then there exists a C^1 vector field $\mathbf{G}$ with $\mathbf{F} =$ curl $\mathbf{G}$.

The proof is outlined in Exercise 20. We should warn you at this point that, unlike the $\mathbf{F}$ in Theorem 7, the vector field $\mathbf{F}$ in Theorem 8 is not allowed to have an exceptional point. For example, the gravitational force field $\mathbf{F} = -(GmM\mathbf{r}/r^3)$ has the property that div $\mathbf{F} = 0$, and yet there is no $\mathbf{G}$ for which $\mathbf{F} =$ curl $\mathbf{G}$ (see Exercise 29). Theorem 8 does not apply, because the gravitational force field $\mathbf{F}$ is not defined at $\mathbf{0} \in \mathbb{R}^3$.

exercises

1. Determine which of the following vector fields $\mathbf{F}$ in the plane is the gradient of a scalar function f. If such an f exists, find it.

(a) $\mathbf{F}(x,y) = x\mathbf{i} + y\mathbf{j}$

(b) $\mathbf{F}(x,y) = xy\mathbf{i} + xy\mathbf{j}$

(c) $\mathbf{F}(x,y) = (x^2 + y^2)\mathbf{i} + 2xy\mathbf{j}$

2. Repeat Exercise 1 for the following vector fields:

(a) $\mathbf{F}(x,y) = (\cos xy - xy \sin xy)\mathbf{i} - (x^2 \sin xy)\mathbf{j}$

(b) $\mathbf{F}(x,y) = (x\sqrt{x^2y^2 + 1})\mathbf{i} + (y\sqrt{x^2y^2 + 1})\mathbf{j}$

(c) $\mathbf{F}(x,y) = (2x \cos y + \cos y)\mathbf{i} - (x^2 \sin y + x \sin y)\mathbf{j}$

3. For each of the following vector fields $\mathbf{F}$, determine (i) if there exists a function g such that $\nabla g = \mathbf{F}$, and (ii) if there exists a vector field $\mathbf{G}$ such that curl $\mathbf{G} = \mathbf{F}$. (It is not necessary to find g or $\mathbf{G}$.)

(a) $\mathbf{F}(x, y, z) = (4xz - x, -4yz, z - 2y)$

(b) $\mathbf{F}(x, y, z) = (e^x \sin y, e^x \cos y, z^2)$

(c) $\mathbf{F}(x, y, z) = (\log(z^2 + 1) + y^2, 2xy, \frac{2xz}{z^2+1})$

(d) $\mathbf{F}(x, y, z) = (x^2 + x \sin z, y \cos z - 2xy, \cos z + \sin z)$

4. For each of the following vector fields $\mathbf{F}$, determine (i) if there exists a function g such that $\nabla g = \mathbf{F}$, and (ii) if there exists a vector field $\mathbf{G}$ such that curl $\mathbf{G} = \mathbf{F}$. (It is not necessary to find g or $\mathbf{G}$.)

(a) $\mathbf{F}(x, y, z) = (e^x \cos y, -e^x \sin y, \pi)$

(b) $\mathbf{F}(x, y, z) = \left(\frac{y}{z^2+4}, \frac{x}{z^2+4}, \frac{-2xyz}{z^4+8z^2+16}\right)$

(c) $\mathbf{F}(x, y, z) = (x^2y^2z^2, ye^x, xy \cos z)$

(d) $\mathbf{F}(x, y, z) = (6z^5y^5, 9x^8z^2, 4x^3y^3)$

5. Show that any two potential functions for a vector field on $\mathbb{R}^3$ differ at most by a constant.

6. (a) Let $\mathbf{F}(x, y) = (xy, y^2)$ and let $\mathbf{c}$ be the path $y = 2x^2$ joining $(0, 0)$ to $(1, 2)$ in $\mathbb{R}^2$. Evaluate $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$.

(b) Does the integral in part (a) depend on the path joining $(0, 0)$ to $(1, 2)$?

7. Let $\mathbf{F}(x, y, z) = (2xyz + \sin x)\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$. Find a function f such that $\mathbf{F} = \nabla f$.

8. Evaluate $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{c}(t) = (\cos^5 t, \sin^3 t, t^4)$, $0 \le t \le \pi$, and $\mathbf{F}$ is as in Exercise 7.

9. If $f(x)$ is a smooth function of one variable, must $\mathbf{F}(x, y) = f(x)\mathbf{i} + f(y)\mathbf{j}$ be a gradient?

10. (a) Show that $\mathbf{F} = -\mathbf{r}/\|\mathbf{r}\|^3$ is the gradient of $f(x, y, z) = 1/r$.

(b) What is the work done by the force $\mathbf{F} = -\mathbf{r}/\|\mathbf{r}\|^3$ in moving a particle from a point $\mathbf{r}_0 \in \mathbb{R}^3$ "to ∞," where $\mathbf{r}(x, y, z) = (x, y, z)$?

11. Let $\mathbf{F}(x, y, z) = xy\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Can there exist a function f such that $\mathbf{F} = \nabla f$?

12. Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ and suppose each F_k satisfies the homogeneity condition

$$F_k(tx, ty, tz) = tF_k(x, y, z), \qquad k = 1, 2, 3.$$

Suppose also $\nabla \times \mathbf{F} = \mathbf{0}$. Prove that $\mathbf{F} = \nabla f$, where

$$2f(x, y, z) = xF_1(x, y, z) + yF_2(x, y, z) + zF_3(x, y, z).$$

[HINT: Use Review Exercise 31, Chapter 2.]

13. Let $\mathbf{F}(x, y, z) = (e^x \sin y)\mathbf{i} + (e^x \cos y)\mathbf{j} + z^2\mathbf{k}$. Evaluate the integral $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{c}(t) = (\sqrt{t}, t^3, \exp \sqrt{t})$, $0 \le t \le 1$.

14. Let a fluid have the velocity field $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$. What is the circulation around the unit circle in the xy plane? Interpret your answer.

15. The mass of the earth is approximately 6×10^{27} g and that of the sun is 330,000 times as much. The gravitational constant is 6.7×10^{-8} cm^3/s$^2 \cdot$ g. The distance of the earth from the sun is about 1.5×10^{12} cm. Compute, approximately, the work necessary to increase the distance of the earth from the sun by 1 cm.

16. (a) Show that $\int_C (x\,dy - y\,dx)/(x^2 + y^2) = 2\pi$, where C is the unit circle.

(b) Conclude that the associated vector field $[-y/(x^2 + y^2)]\mathbf{i} + [x/(x^2 + y^2)]\mathbf{j}$ is not a conservative field.

(c) Show, however, that $\partial P/\partial y = \partial Q/\partial x$. Does this contradict the corollary to Theorem 7? If not, why not?

17. Determine if the following vector fields $\mathbf{F}$ are gradient fields. If there exists a fuction f such that $\nabla f = \mathbf{F}$, find f.

(a) $\mathbf{F}(x, y, z) = (2xyz, x^2z, x^2y)$

(b) $\mathbf{F}(x, y) = (x \cos y, x \sin y)$

(c) $\mathbf{F}(x, y, z) = (x^2e^y, xyz, e^z)$

(d) $\mathbf{F}(x, y) = (2x \cos y, -x^2 \sin y)$

18. Determine if the following vector fields $\mathbf{F}$ are gradient fields. If there exists a function f such that $\nabla f = \mathbf{F}$, find f.

(a) $\mathbf{F}(x, y) = (2x + y^2 - y \sin x, 2xyz + \cos x)$

(b) $\mathbf{F}(x, y, z) = (6x^2z^2, 5x^2y^2, 4y^2z^2)$

(c) $\mathbf{F}(x, y) = (y^3 + 1, 3xy^2 + 1)$

(d) $\mathbf{F}(x, y) = (xe^{(x^2+y^2)} + 2xy, ye^{(x^2+y^2)} + 4y^3z, y^4)$

19. Show that the following vector fields are conservative. Calculate $\int_C \mathbf{F} \cdot d\mathbf{s}$ for the given curve.

(a) $\mathbf{F} = (xy^2 + 3x^2y)\mathbf{i} + (x + y)x^2\mathbf{j}$; C is the curve consisting of line segments from $(1, 1)$ to $(0, 2)$ to $(3, 0)$.

(b) $\mathbf{F} = \dfrac{2x}{y^2 + 1}\mathbf{i} - \dfrac{2y(x^2 + 1)}{(y^2 + 1)^2}\mathbf{j}$; C is parametrized by $x = t^3 - 1$, $y = t^6 - t$, $0 \le t \le 1$.

(c) $\mathbf{F} = [\cos(xy^2) - xy^2 \sin(xy^2)]\mathbf{i} - 2x^2y \sin(xy^2)\mathbf{j}$; C is the curve (e^t, e^{t+1}), $-1 \le t \le 0$.

20. Prove Theorem 8. [HINT: Define $\mathbf{G} = G_1\mathbf{i} + G_2\mathbf{j} + G_3\mathbf{k}$ by

$$G_1(x, y, z) = \int_0^z F_2(x, y, t)\, dt - \int_0^y F_3(x, t, 0)\, dt$$
$$G_2(x, y, z) = -\int_0^z F_1(x, y, t)\, dt$$

and $G_3(x, y, z) = 0$.]

21. Is each of the following vector fields the curl of some other vector field? If so, find the vector field.

(a) $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

(b) $\mathbf{F} = (x^2 + 1)\mathbf{i} + (z - 2xy)\mathbf{j} + y\mathbf{k}$

22. Let $\mathbf{F} = xz\mathbf{i} - yz\mathbf{j} + y\mathbf{k}$. Verify that $\nabla \cdot \mathbf{F} = 0$. Find a $\mathbf{G}$ such that $\mathbf{F} = \nabla \times \mathbf{G}$.

23. Repeat Exercise 22 for $\mathbf{F} = y^2\mathbf{i} + z^2\mathbf{j} + x^2\mathbf{k}$.

24. Let $\mathbf{F} = xe^y\mathbf{i} - (x \cos z)\mathbf{j} - ze^y\mathbf{k}$. Find a $\mathbf{G}$ such that $\mathbf{F} = \nabla \times \mathbf{G}$.

25. Let $\mathbf{F} = (x \cos y)\mathbf{i} - (\sin y)\mathbf{j} + (\sin x)\mathbf{k}$. Find a $\mathbf{G}$ such that $\mathbf{F} = \nabla \times \mathbf{G}$.

26. By using different paths from $(0, 0, 0)$ to (x, y, z), show that the function f defined in the proof of Theorem 7 for "condition (ii) implies condition (iii)" satisfies $\partial f/\partial x = F_1$ and $\partial f/\partial y = F_2$.

27. Let $\mathbf{F}$ be the vector field on $\mathbb{R}^3$ given by $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$.

(a) Show that $\mathbf{F}$ is rotational, that is, $\mathbf{F}$ is not irrotational.

(b) Suppose $\mathbf{F}$ represents the velocity vector field of a fluid. Show that if we place a cork in this fluid, it will revolve in a plane parallel to the xy plane, in a circular trajectory about the z axis.

(c) In what direction does the cork revolve?

28. Let $\mathbf{G}$ be the vector field on $\mathbb{R}^3\backslash\{z \text{ axis}\}$ defined by

$$\mathbf{G} = \frac{-y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j}.$$

(a) Show that $\mathbf{G}$ is irrotational.

(b) Show that the result of Exercise 27(b) holds for $\mathbf{G}$ also.

(c) How can we resolve the fact that the trajectories of $\mathbf{F}$ and $\mathbf{G}$ are both the same (circular about the z axis) yet $\mathbf{F}$ is rotational and $\mathbf{G}$ is not? [HINT: The property of being rotational is a local condition, that is, a property of the fluid in the neighborhood of a point.]

29. Let $\mathbf{F} = -(GmM\mathbf{r}/r^3)$ be the gravitational force field defined on $\mathbb{R}^3\backslash\{\mathbf{0}\}$.

(a) Show that $\operatorname{div} \mathbf{F} = 0$.

(b) Show that $\mathbf{F} \neq \operatorname{curl} \mathbf{G}$ for any C^1 vector field $\mathbf{G}$ on $\mathbb{R}^3\backslash\{\mathbf{0}\}$.

8.4 Gauss' Theorem

Gauss' theorem states that the flux of a vector field out of a closed surface equals the integral of the divergence of that vector field over the volume enclosed by the surface. The result parallels Stokes' theorem and Green's theorem in that it relates an integral over a closed geometric object (curve or surface) to an integral over a contained region (surface or volume).

Elementary Regions and Their Boundaries

We shall begin by asking you to review the various elementary regions in space that were introduced when we considered the volume integral; these regions are illustrated in Figures 5.5.2 and 5.5.4. As these figures indicate, the boundary of an elementary region in $\mathbb{R}^3$ is a surface made up of a finite number (at most six, at least two) of surfaces that can be described as graphs of functions from $\mathbb{R}^2$ to $\mathbb{R}$. This kind of surface is called a *closed surface.* The surfaces $S_1, S_2, \ldots, S_N$ composing such a closed surface are called its *faces.*

example 1 The cube in Figure 8.4.1(a) is an elementary region, and in fact a symmetric elementary region, with six rectangles composing its boundary. The sphere in Figure 8.4.1(b) is the boundary of a solid ball, which is also a symmetric elementary region.