exercises

1. The helicoid can be described by

$$\Phi(u, v) = (u \cos v, u \sin v, bv), \text{ where } b \neq 0.$$

Show that $H = 0$ and that $K = -b^2/(b^2 + u^2)^2$. In Figures 7.7.1 and 7.7.5, we see that the helicoid is actually a soap film surface. Surfaces in which $H = 0$ are called ***minimal surfaces.***

2. Consider the saddle surface $z = xy$. Show that

$$K = \frac{-1}{(1 + x^2 + y^2)^2},$$

and that

$$H = \frac{-xy}{(1 + x^2 + y^2)^{3/2}}.$$

3. Show that $\Phi(u, v) = (u, v, \log \cos v - \log \cos u)$ has mean curvature zero (and is thus a minimal surface; see Exercise 1).

4. Find the Gauss curvature of the elliptic paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}.$$

5. Find the Gauss curvature of the hyperbolic paraboloid

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}.$$

6. Compute the Gauss curvature of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1.$$

7. After finding K in Exercise 6, integrate K to show that:

$$\frac{1}{2\pi} \iint_S K \, dA = 2.$$

8. Find the curvature K of:

(a) the cylinder $\Phi(u, v) = (2 \cos v, 2 \sin v, u)$

(b) the surface $\Phi(u, v) = (u, v, u^2)$

9. Show that Enneper's surface

$$\Phi(u, v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + u^2 v, u^2 - v^2\right)$$

is a minimal surface ($H = 0$).

10. Consider the torus T given in Exercise 4, Section 7.4. Compute its Gauss curvature and verify the theorem of Gauss–Bonnet. [Hint: Show that $\|\mathbf{T}_\theta \times \mathbf{T}_\phi\|^2 = (R + \cos \phi)^2$ and $K = \cos \phi/(R + \cos \phi)$.]

11. Let $\Phi(u, v) = (u, h(u) \cos v, h(u) \sin v)$, $h > 0$, be a surface of revolution. Show that $K = -h''/h\{1 + (h')^2\}^2$.

12. A parametrization Φ of a surface S is said to be ***conformal*** (see Section 7.4), provided that $E = G$, $F = 0$. Assume that Φ conformally parametrizes S.[19] Show that if H and K vanish identically, then S must be part of a plane in $\mathbb{R}^3$.

review exercises for chapter 7

1. Integrate $f(x, y, z) = xyz$ along the following paths:

(a) $\mathbf{c}(t) = (e^t \cos t, e^t \sin t, 3)$, $0 \le t \le 2\pi$

(b) $\mathbf{c}(t) = (\cos t, \sin t, t)$, $0 \le t \le 2\pi$

(c) $\mathbf{c}(t) = \frac{3}{2}t^2\mathbf{i} + 2t^2\mathbf{j} + t\mathbf{k}$, $0 \le t \le 1$

(d) $\mathbf{c}(t) = t\mathbf{i} + (1/\sqrt{2})t^2\mathbf{j} + \frac{1}{3}t^3\mathbf{k}$, $0 \le t \le 1$

2. Compute the integral of f along the path $\mathbf{c}$ in each of the following cases:

(a) $f(x, y, z) = x + y + yz$; $\mathbf{c}(t) = (\sin t, \cos t, t)$, $0 \le t \le 2\pi$

(b) $f(x, y, z) = x + \cos^2 z$; $\mathbf{c}(t) = (\sin t, \cos t, t)$, $0 \le t \le 2\pi$

[19] Gauss proved that conformal parametrization of a surface always exists. The result of this exercise remains valid even if Φ is not conformal, but the proof is more difficult.

(c) $f(x, y, z) = x + y + z$; $\mathbf{c}(t) = (t, t^2, \frac{2}{3}t^3)$, $0 \le t \le 1$

3. Compute each of the following line integrals:

(a) $\int_C (\sin \pi x)\, dy - (\cos \pi y)\, dz$, where C is the triangle whose vertices are $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, in that order

(b) $\int_C (\sin z)\, dx + (\cos z)\, dy - (xy)^{1/3}\, dz$, where C is the path $\mathbf{c}(\theta) = (\cos^3 \theta, \sin^3 \theta, \theta)$, $0 \le \theta \le 7\pi/2$

4. If $\mathbf{F}(\mathbf{x})$ is orthogonal to $\mathbf{c}'(t)$ at each point on the curve $\mathbf{x} = \mathbf{c}(t)$, what can you say about $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$?

5. Find the work done by the force $\mathbf{F}(x, y) = (x^2 - y^2)\mathbf{i} + 2xy\mathbf{j}$ in moving a particle counterclockwise around the square with corners $(0, 0)$, $(a, 0)$, (a, a), $(0, a)$, $a > 0$.

6. A ring in the shape of the curve $x^2 + y^2 = a^2$ is formed of thin wire weighing $|x| + |y|$ grams per unit length at (x, y). Find the mass of the ring.

7. Find a parametrization for each of the following surfaces:

(a) $x^2 + y^2 + z^2 - 4x - 6y = 12$

(b) $2x^2 + y^2 + z^2 - 8x = 1$

(c) $4x^2 + 9y^2 - 2z^2 = 8$

8. Find the area of the surface defined by $\mathbf{\Phi}: (u, v) \mapsto (x, y, z)$, where

$$x = h(u, v) = u + v, \qquad y = g(u, v) = u,$$
$$z = f(u, v) = v;$$

$0 \le u \le 1$, $0 \le v \le 1$. Sketch.

9. Write a formula for the surface area of $\mathbf{\Phi}: (r, \theta) \mapsto (x, y, z)$, where

$$x = r \cos \theta, \qquad y = 2r \sin \theta, \qquad z = r;$$

$0 \le r \le 1$, $0 \le \theta \le 2\pi$. Describe the surface.

10. Suppose $z = f(x, y)$ and $(\partial f/\partial x)^2 + (\partial f/\partial y)^2 = c$, $c > 0$. Show that the area of the graph of f lying over a region D in the xy plane is $\sqrt{1 + c}$ times the area of D.

11. Compute the integral of $f(x, y, z) = x^2 + y^2 + z^2$ over the surface in Review Exercise 8.

12. Find $\iint_S f\, dS$ in each of the following cases:

(a) $f(x, y, z) = x$; S is the part of the plane $x + y + z = 1$ in the positive octant defined by $x \ge 0$, $y \ge 0$, $z \ge 0$

(b) $f(x, y, z) = x^2$; S is the part of the plane $x = z$ inside the cylinder $x^2 + y^2 = 1$

(c) $f(x, y, z) = x$; S is the part of the cylinder $x^2 + y^2 = 2x$ with $0 \le z \le \sqrt{x^2 + y^2}$

13. Compute the integral of $f(x, y, z) = xyz$ over the rectangle with vertices $(1, 0, 1)$, $(2, 0, 0)$, $(1, 1, 1)$, and $(2, 1, 0)$.

14. Compute the integral of $x + y$ over the surface of the unit sphere.

15. Compute the surface integral of x over the triangle with vertices $(1, 1, 1)$, $(2, 1, 1)$, and $(2, 0, 3)$.

16. A paraboloid of revolution S is parametrized by $\mathbf{\Phi}(u, v) = (u \cos v, u \sin v, u^2)$, $0 \le u \le 2$, $0 \le v \le 2\pi$.

(a) Find an equation in x, y, and z describing the surface.

(b) What are the geometric meanings of the parameters u and v?

(c) Find a unit vector orthogonal to the surface at $\mathbf{\Phi}(u, v)$.

(d) Find the equation for the tangent plane at $\mathbf{\Phi}(u_0, v_0) = (1, 1, 2)$ and express your answer in the following two ways:

(i) parametrized by u and v; and

(ii) in terms of x, y, and z.

(e) Find the area of S.

17. Let $f(x, y, z) = xe^y \cos \pi z$.

(a) Compute $\mathbf{F} = \nabla f$.

(b) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{c}(t) = (3 \cos^4 t, 5 \sin^7 t, 0)$, $0 \le t \le \pi$.

18. Let $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is the upper hemisphere of the unit sphere $x^2 + y^2 + z^2 = 1$.

19. Let $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Evaluate $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{c}(t) = (e^t, t, t^2)$, $0 \le t \le 1$.

20. Let $\mathbf{F} = \nabla f$ for a given scalar function. Let $\mathbf{c}(t)$ be a closed curve, that is, $\mathbf{c}(b) = \mathbf{c}(a)$. Show that $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = 0$.

21. Consider the surface $\mathbf{\Phi}(u, v) = (u^2 \cos v, u^2 \sin v, u)$. Compute the unit normal at $u = 1$, $v = 0$. Compute the equation of the tangent plane at this point.

22. Let S be the part of the cone $z^2 = x^2 + y^2$ with z between 1 and 2 oriented by the normal pointing out of the cone. Compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = (x^2, y^2, z^2)$.

23. Let $\mathbf{F} = x\mathbf{i} + x^2\mathbf{j} + yz\mathbf{k}$ represent the velocity field of a fluid (velocity measured in meters per second). Compute how many cubic meters of fluid per second are crossing the xy plane through the square $0 \le x \le 1, 0 \le y \le 1$.

24. Show that the surface area of the part of the sphere $x^2 + y^2 + z^2 = 1$ lying above the rectangle $[-a, a] \times [-a, a]$, where $2a^2 < 1$, in the xy plane is

$$A = 2\int_{-a}^{a} \sin^{-1}\left(\frac{a}{\sqrt{1 - x^2}}\right) dx.$$

25. Let S be a surface and C a closed curve bounding S. Verify the equality

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{s}$$

if $\mathbf{F}$ is a gradient field (use Review Exercise 20).

26. Calculate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = (x, y, -y)$ and S is the cylindrical surface defined by $x^2 + y^2 = 1$, $0 \le z \le 1$, with normal pointing out of the cylinder.

27. Let S be the portion of the cylinder $x^2 + y^2 = 4$ between the planes $z = 0$ and $z = x + 3$. Compute the following:

(a) $\iint_S x^2\, dS$

(b) $\iint_S y^2\, dS$

(c) $\iint_S z^2\, dS$

28. Let Γ be the curve of intersection of the plane $z = ax + by$, with the cylinder $x^2 + y^2 = 1$. Find all values of the real numbers a and b such that $a^2 + b^2 = 1$ and

$$\int_\Gamma y\, dx + (z - x)\, dy - y\, dz = 0.$$

29. A circular helix that lies on the cylinder $x^2 + y^2 = R^2$ with pitch p may be described parametrically by

$$x = R\cos\theta, \qquad y = R\sin\theta, \qquad z = p\theta, \qquad \theta \ge 0.$$

A particle slides under the action of gravity (which acts parallel to the z axis) without friction along the helix. If the particle starts out at the height $z_0 > 0$, then when it reaches the height z, $0 \le z < z_0$, along the helix, its speed is given by

$$\frac{ds}{dt} = \sqrt{(z_0 - z)2g},$$

where s is arc length along the helix, g is the constant of gravity, and t is time.

(a) Find the length of the part of the helix between the planes $z = z_0$ and $z = z_1$, $0 \le z_1 < z_0$.

(b) Compute the time T_0 it takes the particle to reach the plane $z = 0$.