the same formula we derived (in Chapter 2) for the plane tangent to S at the point $(x_0, y_0, z_0) \in S$.

It is also useful to consider piecewise smooth surfaces, that is, surfaces composed of a certain number of images of smooth parametrized surfaces. For example, the surface of a cube in $\mathbb{R}^3$ is such a surface. These surfaces are considered in Section 7.4.

example 5 Find a parametrization for the hyperboloid of one sheet:

$$x^2 + y^2 - z^2 = 1.$$

solution Because x and y appear in the combination $x^2 + y^2$, the surface is invariant under rotation about the z axis, and so it is natural to write

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Then $x^2 + y^2 - z^2 = 1$ becomes $r^2 - z^2 = 1$. This we can conveniently parametrize by[9]

$$r = \cosh u, \qquad z = \sinh u.$$

Thus, a parametrization is

$$x = (\cosh u)(\cos\theta), \qquad y = (\cosh u)(\sin\theta), \qquad z = \sinh u,$$

where $0 \le \theta < 2\pi$, $-\infty < u < \infty$. ▲

exercises

In Exercises 1 to 3, find an equation for the plane tangent to the given surface at the specified point.

1. $x = 2u, \quad y = u^2 + v, \quad z = v^2, \quad at\ (0, 1, 1)$

2. $x = u^2 - v^2, \quad y = u + v, \quad z = u^2 + 4v,$ $at\ (-\frac{1}{4}, \frac{1}{2}, 2)$

3. $x = u^2, \quad y = u\sin e^v, \quad z = \frac{1}{3}u\cos e^v,$ $at\ (13, -2, 1)$

4. At what points are the surfaces in Exercises 1 and 2 regular?

*In Exercises 5 and 6, find all points (u_0, v_0), where $S = \Phi(u_0, v_0)$ is **not** smooth (regular).*

5. $\Phi(u, v) = (u^2 - v^2, u^2 + v^2, v)$

6. $\Phi(u, v) = (u - v, u + v, 2uv)$

7. Match the following parameterizations to the surfaces shown in the figures.

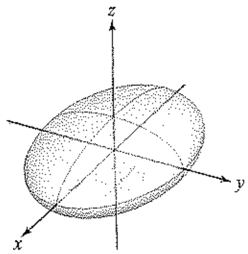
   (a) $\Phi(u, v) = ((2\sqrt{1 + u^2})\cos v, (2\sqrt{1 + u^2})\sin v, u)$

   (b) $\Phi(u, v) = (3\cos u \sin v, 2\sin u \sin v, \cos v)$

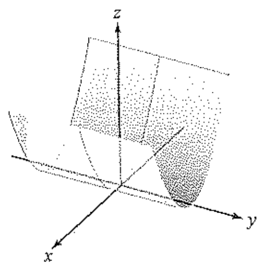
   (c) $\Phi(u, v) = (u, v, u^2)$

   (d) $\Phi(u, v) = (u\cos v, u\sin v, u)$

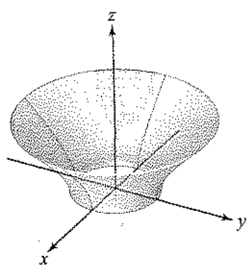
[9]Recall from one-variable calculus that $\cosh u = (e^u + e^{-u})/2$ and $\sinh u = (e^u - e^{-u})/2$. We easily verify from these definitions that $\cosh^2 u - \sinh^2 u = 1$.



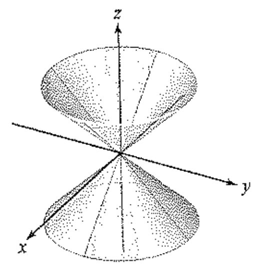
(i)



(ii)



(iii)



(iv)

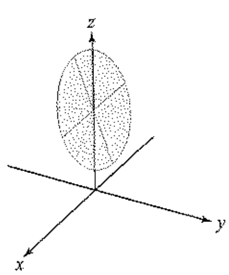
8. Match the following parametrizations to the surfaces shown in the figures.

(a) $\Phi(u, v) = (u\cos v, u\sin v, 4 - u\cos v - u\sin v)$; $u \in [0, 1]$, $v \in [0, 2\pi]$

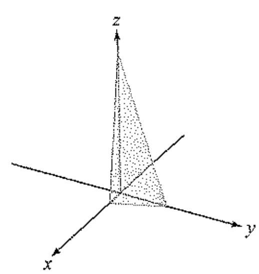
(b) $\Phi(u, v) = (u\cos v, u\sin v, 4 - u^2)$

(c) $\Phi(u, v) = (u, v, \frac{1}{3}(12 - 8u - 3v))$

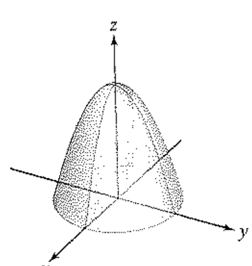
(d) $\Phi(u, v) = ((u^2 + 6u + 11)\cos v, u, (u^2 + 6u + 11)\sin v)$



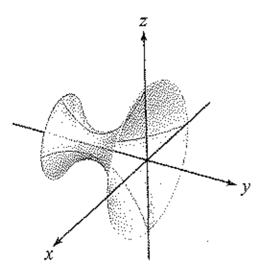
(i)



(ii)



(iii)



(iv)

9. Find an expression for a unit vector normal to the surface

$$x = \cos v \sin u, \qquad y = \sin v \sin u, \qquad z = \cos u$$

at the image of a point (u, v) for u in $[0, \pi]$ and v in $[0, 2\pi]$. Identify this surface.

10. Repeat Exercise 9 for the surface

$$x = 3\cos\theta\sin\phi, \qquad y = 2\sin\theta\sin\phi, \qquad z = \cos\phi$$

for θ in $[0, 2\pi]$ and ϕ in $[0, \pi]$.

11. Repeat Exercise 9 for the surface

$$x = \sin v, \qquad y = u, \qquad z = \cos v$$

for $0 \le v \le 2\pi$ and $-1 \le u \le 3$.

12. Repeat Exercise 9 for the surface

$$x = (2 - \cos v)\cos u, \quad y = (2 - \cos v)\sin u, \quad z = \sin v$$

for $-\pi \le u \le \pi$, $-\pi \le v \le \pi$. Is this surface regular?

13. (a) Develop a formula for the plane tangent to the surface $x = h(y, z)$.

(b) Obtain a similar formula for $y = k(x, z)$.

14. Find the equation of the plane tangent to the surface $x = u^2$, $y = v^2$, $z = u^2 + v^2$ at the point $u = 1$, $v = 1$.

15. Find a parametrization of the surface $z = 3x^2 + 8xy$ and use it to find the tangent plane at $x = 1$, $y = 0$, $z = 3$. Compare your answer with that using graphs.

16. Find a parametrization of the surface $x^3 + 3xy + z^2 = 2$, $z > 0$, and use it to find the tangent plane at the point $x = 1$, $y = 1/3$, $z = 0$. Compare your answer with that using level sets.

17. Consider the surface in $\mathbb{R}^3$ parametrized by

$$\Phi(r, \theta) = (r\cos\theta, r\sin\theta, \theta), \quad 0 \le r \le 1 \text{ and } 0 \le \theta \le 4\pi.$$

(a) Sketch and describe the surface.

(b) Find an expression for a unit normal to the surface.

(c) Find an equation for the plane tangent to the surface at the point (x_0, y_0, z_0).

(d) If (x_0, y_0, z_0) is a point on the surface, show that the horizontal line segment of unit length from the z axis through (x_0, y_0, z_0) is contained in the surface and in the plane tangent to the surface at (x_0, y_0, z_0).

18. Given a sphere of radius 2 centered at the origin, find the equation for the plane that is tangent to it at the point $(1, 1, \sqrt{2})$ by considering the sphere as:

(a) a surface parametrized by $\Phi(\theta, \phi) = (2\cos\theta\sin\phi, 2\sin\theta\sin\phi, 2\cos\phi)$;

(b) a level surface of $f(x, y, z) = x^2 + y^2 + z^2$; and

(c) the graph of $g(x, y) = \sqrt{4 - x^2 - y^2}$.

19. (a) Find a parametrization for the hyperboloid $x^2 + y^2 - z^2 = 25$.

(b) Find an expression for a unit normal to this surface.

(c) Find an equation for the plane tangent to the surface at $(x_0, y_0, 0)$, where $x_0^2 + y_0^2 = 25$.

(d) Show that the lines $(x_0, y_0, 0) + t(-y_0, x_0, 5)$ and $(x_0, y_0, 0) + t(y_0, -x_0, 5)$ lie in the surface *and* in the tangent plane found in part (c).

20. A parametrized surface is described by a differentiable function $\mathbf{\Phi}: \mathbb{R}^2 \to \mathbb{R}^3$. According to Chapter 2, the derivative should give a linear approximation that yields a representation of the tangent plane. This exercise demonstrates that this is indeed the case.

(a) Assuming $\mathbf{T}_u \times \mathbf{T}_v \neq \mathbf{0}$, show that the range of the linear transformation $\mathbf{D\Phi}(u_0, v_0)$ is the plane spanned by $\mathbf{T}_u$ and $\mathbf{T}_v$. [Here $\mathbf{T}_u$ and $\mathbf{T}_v$ are evaluated at (u_0, v_0).]

(b) Show that $\mathbf{w} \perp (\mathbf{T}_u \times \mathbf{T}_v)$ if and only if $\mathbf{w}$ is in the range of $\mathbf{D\Phi}(u_0, v_0)$.

(c) Show that the tangent plane as defined in this section is the same as the "parametrized plane"

$$(u, v) \mapsto \mathbf{\Phi}(u_0, v_0) + \mathbf{D\Phi}(u_0, v_0)\begin{bmatrix} u - u_0 \\ v - v_0 \end{bmatrix}.$$

21. Consider the surfaces $\mathbf{\Phi}_1(u, v) = (u, v, 0)$ and $\mathbf{\Phi}_2(u, v) = (u^3, v^3, 0)$.

(a) Show that the image of $\mathbf{\Phi}_1$ and of $\mathbf{\Phi}_2$ is the xy plane.

(b) Show that $\mathbf{\Phi}_1$ describes a regular surface, yet $\mathbf{\Phi}_2$ does not. Conclude that the notion of regularity of a surface S depends on the existence of at least one regular parametrization for S.

(c) Prove that the tangent plane of S is well defined independently of the regular (one-to-one) parametrization (you will need to use the inverse function theorem from Section 3.5).

(d) After these remarks, do you think you can find a regular parametrization of the cone of Figure 7.3.7?

22. The image of the parametrization

$$\begin{aligned}\Phi(u, v) &= (x(u, v), y(u, v), z(u, v)) \\ &= (a\sin u\cos v, b\sin u\sin v, c\cos u)\end{aligned}$$

with $b < a$, $0 \le u \le \pi$, $0 \le v \le 2\pi$ parametrizes an ellipsoid.

(a) Show that all points in the image of Φ satisfy:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

(the Cartesian equation of an ellipsoid).

(b) Show that the image surface is regular at all points.

23. The image of the parametrization

$$\begin{aligned}\Phi(u, v) &= (x(u, v), y(u, v), z(u, v)) \\ &= ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u)\end{aligned}$$

with $0 \le u, v \le 2\pi$, $0 < r < 1$ parametrizes a torus (or doughnut) S.

(a) Show that all points in the image (x, y, z) satisfy:

$$(\sqrt{x^2 + y^2} - R)^2 + z^2 = r^2.$$

(b) Show that the image surface is regular at all points.

24. Let $\mathbf{\Phi}$ be a regular surface at (u_0, v_0); that is, $\mathbf{\Phi}$ is of class C^1 and $\mathbf{T}_u \times \mathbf{T}_v \neq \mathbf{0}$ at (u_0, v_0).

(a) Use the implicit function theorem (Section 3.5) to show that the image of $\mathbf{\Phi}$ *near* (u_0, v_0) is the graph of a C^1 function of two variables. If the z component of $\mathbf{T}_u \times \mathbf{T}_v$ is nonzero, we can write it as $z = f(x, y)$.

(c) Show that the tangent plane at $\mathbf{\Phi}(u_0, v_0)$ defined by the plane spanned by $\mathbf{T}_u$ and $\mathbf{T}_v$ coincides with the tangent plane of the graph of $z = f(x, y)$ at this point.

7.4 Area of a Surface

Before proceeding to general surface integrals, let us first consider the problem of computing the area of a surface, just as we considered the problem of finding the arc length of a curve before discussing path integrals.