The formal statement of this fact is known as the *Fary–Milnor* theorem. Legend has it that John Milnor, a contemporary of John Nash's[1] at Princeton University, was asleep in a math class as the professor wrote three *unsolved* knot theory problems on the blackboard. At the end of the class, Milnor (still an undergraduate) woke up and, thinking the blackboard problems were assigned as homework, quickly wrote them down. The following week he turned in the solution to all three problems—one of which was a proof of the Fary–Milnor theorem! Some years later, he was appointed a professor at Princeton, and in 1962 he was awarded (albeit for other work) a Fields medal, mathematics' highest honor, generally regarded as the mathematical Nobel Prize.

exercises

In Exercises 1 to 4, find an appropriate parametrization for the given piecewise-smooth curve in $\mathbb{R}^2$, with the implied orientation.

1. The curve C, which goes along the circle of radius 3, from the point $(3, 0)$ to the point $(-3, 0)$, and then in a straight line along the x-axis back to $(3, 0)$

2. The curve C, which goes along $y = x^2$ from the point $(0, 0)$ to the point $(2, 4)$, then in a straight line from $(2, 4)$ to $(0, 4)$, and then along the y-axis back to $(0, 0)$

3. The curve C, which goes along $y = \sin x$ from the point $(0, 0)$ to the point $(\pi, 0)$, and then along the x-axis back to $(0, 0)$

4. The closed curve C described by the ellipse

$$\frac{(x-2)^2}{4} + \frac{(y-3)^2}{9} = 1$$

oriented counterclockwise

In Exercises 5 to 8, find an appropriate parametrization for the given piecewise-smooth curve in $\mathbb{R}^3$.

5. The intersection of the plane $z = 3$ with the elliptical cylinder

$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

6. The triangle formed by traveling from the point $(1, 2, 3)$ to $(0, -2, 1)$, to $(6, 4, 2)$, and back to $(1, 2, 3)$

7. The intersection of the surfaces $y = x$ and $z = x^3$, from the point $(-3, -3, 9)$ to $(2, 2, 4)$

8. The intersection of the cylinder $y^2 + z^2 = 1$ and the plane $z = x$

9. Let $f(x, y, z) = y$ and $\mathbf{c}(t) = (0, 0, t)$, $0 \le t \le 1$. Prove that $\int_{\mathbf{c}} f\, ds = 0$.

10. Evaluate the following path integrals $\int_{\mathbf{c}} f(x, y, z)\, ds$, where

(a) $f(x, y, z) = x + y + z$ and $\mathbf{c}\colon t \mapsto (\sin t, \cos t, t)$, $t \in [0, 2\pi]$

(b) $f(x, y, z) = \cos z$, $\mathbf{c}$ as in part (a)

11. Evaluate the following path integrals $\int_{\mathbf{c}} f(x, y, z)\, ds$, where

(a) $f(x, y, z) = \exp \sqrt{z}$, and $\mathbf{c}\colon t \mapsto (1, 2, t^2)$, $t \in [0, 1]$

(b) $f(x, y, z) = yz$, and $\mathbf{c}\colon t \mapsto (t, 3t, 2t)$, $t \in [1, 3]$

12. Evaluate the integral of $f(x, y, z)$ along the path $\mathbf{c}$, where

(a) $f(x, y, z) = x \cos z$, $\mathbf{c}\colon t \mapsto t\mathbf{i} + t^2\mathbf{j}$, $t \in [0, 1]$

(b) $f(x, y, z) = (x + y)/(y + z)$, and $\mathbf{c}\colon t \mapsto \left(t, \frac{2}{3}t^{3/2}, t\right)$, $t \in [1, 2]$

13. Let $f\colon \mathbb{R}^3 \setminus \{xz \text{ plane}\} \to \mathbb{R}$ be defined by $f(x, y, z) = 1/y^3$. Evaluate $\int_{\mathbf{c}} f(x, y, z)\, ds$, where $\mathbf{c}\colon [1, e] \to \mathbb{R}^3$ is given by $\mathbf{c}(t) = (\log t)\mathbf{i} + t\mathbf{j} + 2\mathbf{k}$.

[1] John Nash is the subject of Sylvia Nasar's best-selling biography, *A Beautiful Mind*, a fictionalized version of which was made into a movie in 2001.

14. (a) Show that the path integral of $f(x, y)$ along a path given in polar coordinates by $r = r(\theta)$, $\theta_1 \leq \theta \leq \theta_2$, is

$$\int_{\theta_1}^{\theta_2} f(r\cos\theta, r\sin\theta)\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

(b) Compute the arc length of the path $r = 1 + \cos\theta$, $0 \leq \theta \leq 2\pi$.

15. Let $f(x, y) = 2x - y$, and consider the path $x = t^4$, $y = t^4$, $-1 \leq t \leq 1$.

(a) Compute the integral of f along this path and interpret the answer geometrically.

(b) Evaluate the arc-length function $s(t)$ and redo part (a) in terms of s (you may wish to consult Exercise 2, Section 4.2).

Exercises 16 to 19 are concerned with the application of the path integral to the problem of defining the average value of a scalar function along a path. Define the number

$$\frac{\int_{\mathbf{c}} f(x, y, z)\, ds}{l(\mathbf{c})}$$

to be the ***average value*** *of f along $\mathbf{c}$. Here $l(\mathbf{c})$ is the length of the path:*

$$l(\mathbf{c}) = \int_{\mathbf{c}} \|\mathbf{c}'(t)\|\, dt.$$

(This is analogous to the average of a function over a region defined in Section 6.3.)

16. (a) Justify the formula $[\int_{\mathbf{c}} f(x, y, z)\, ds]/l(\mathbf{c})$ for the average value of f along $\mathbf{c}$ using Riemann sums.

(b) Show that the average value of f along $\mathbf{c}$ in Example 1 is $(1 + \frac{4}{3}\pi^2)$.

(c) In Exercise 10(a) and (b) above, find the average value of f over the given curves.

17. Find the average y coordinate of the points on the semicircle parametrized by $\mathbf{c}$: $[0, \pi] \to \mathbb{R}^3$, $\theta \mapsto (0, a\sin\theta, a\cos\theta)$; $a > 0$.

18. Suppose the semicircle in Exercise 17 is made of a wire with a uniform density of 2 grams per unit length.

(a) What is the total mass of the wire?

(b) Where is the center of mass of this configuration of wire? (Consult Section 6.3.)

19. Let $\mathbf{c}$ be the path given by $\mathbf{c}(t) = (t^2, t, 3)$ for $t \in [0, 1]$.

(a) Find $l(\mathbf{c})$, the length of the path.

(b) Find the average y coordinate along the path $\mathbf{c}$.

20. Show that the path integral of a function $f(x, y)$ over a path C given by the graph of $y = g(x)$, $a \leq x \leq b$ is given by:

$$\int_C f\, ds = \int_a^b f(x, g(x))\sqrt{1 + [g'(x)]^2}\, dx$$

Conclude that if $g : [a, b] \to \mathbb{R}$ is piecewise continuously differentiable, then the length of the graph of g on $[a, b]$ is given by:

$$\int_C f\, ds = \int_a^b \sqrt{1 + g'(x)^2}\, dx.$$

21. If g: $[a, b] \to \mathbb{R}$ is piecewise continuously differentiable, let the *length of the graph* of g on $[a, b]$ be defined as the length of the path $t \mapsto (t, g(t))$ for $t \in [a, b]$. Show that the length of the graph of g on $[a, b]$ is

$$\int_a^b \sqrt{1 + [g'(x)]^2}\, dx.$$

22. Use Exercise 21 to find the length of the graph of $y = \log x$ from $x = 1$ to $x = 2$.

23. Use Exercise 20 to evaluate the path integral of $f(x, y) = y$ over the graph of the semicircle $y = \sqrt{1 - x^2}$, $-1 \leq x \leq 1$.

24. Compute the path integral of $f(x, y) = y^2$ over the graph $y = e^x$, $0 \leq x \leq 1$.

25. Compute the path integral of $f(x, y, z) = xyz$ over the path $c(t) = (\cos t, \sin t, t)$, $0 \leq t \leq \frac{\pi}{2}$.

26. Find the mass of a wire formed by the intersection of the sphere $x^2 + y^2 + z^2 = 1$ and the plane $x + y + z = 0$ if the density at (x, y, z) is given by $\rho(x, y, z) = x^2$ grams per unit length of wire.

27. Evaluate $\int_{\mathbf{c}} f\, ds$, where $f(x, y, z) = z$ and $\mathbf{c}(t) = (t \cos t, t \sin t, t)$ for $0 \leq t \leq t_0$.

28. Write the following limit as a path integral of $f(x, y, z) = xy$ over some path $\mathbf{c}$ on $[0, 1]$ and evaluate:

$$\lim_{N\to\infty} \sum_{i=1}^{N-1} t_i^2 \left(t_{i+1}^2 - t_i^2\right),$$

where $t_1, \ldots, t_N$ is a partition of $[0, 1]$.

29. Consider paths that connect the points $A = (0, 1)$ and $B = (1, 0)$ in the xy plane, as in Figure 7.1.5.[2]

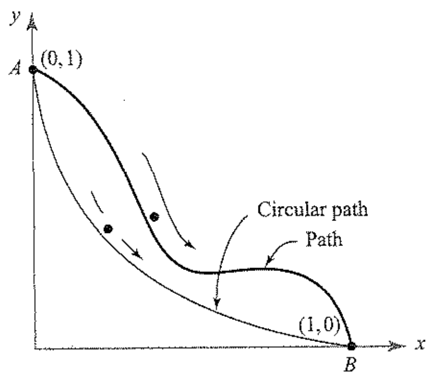

figure **7.1.5** A curve joining the points A and B.

Galileo contemplated the following question: Does a bead falling under the influence of gravity from a point A to a point B along a curve do so in *the least possible time* if that curve is a circular arc? For any given path, the time of transit T is a path integral

$$T = \int \frac{dt}{v},$$

where the bead's velocity is $v = \sqrt{2gy}$, where g is the gravitational constant. In 1697, Johann Bernoulli challenged the mathematical world to find the path in which the bead would roll from A to B in the least time. This solution would determine whether Galileo's considerations had been correct.

(a) Calculate T for the straight-line path $y = 1 - x$.

(b) Write a formula for T for Galileo's circular path, given by $(x - 1)^2 + (y - 1)^2 = 1$.

Incidentally, Newton was the first to send his solution [which turned out to be a cycloid—the same curve (inverted) that we studied in Section 2.4, Example 4], but he did so anonymously. Bernoulli was not fooled, however. When he received the solution, he immediately knew its author, exclaiming, "I know the Lion from his paw." While the solution of this problem is a cycloid, it is known in the literature as the *brachistrochrone*. This was the beginning of the important field called the *calculus of variations*.

7.2 Line Integrals

We now consider the problem of integrating a *vector field* along a path. We will begin by considering the notion of *work* to motivate the general definition.

Work Done by Force Fields

If $\mathbf{F}$ is a force field in space, then a test particle (for example, a small unit charge in an electric force field or a unit mass in a gravitational field) will experience the force $\mathbf{F}$. Suppose the particle moves along the image of a path $\mathbf{c}$ while being acted upon by $\mathbf{F}$. A fundamental concept is the *work done* by $\mathbf{F}$ on the particle as it traces out the path $\mathbf{c}$. If $\mathbf{c}$ is a straight-line displacement given by the vector $\mathbf{d}$ and if $\mathbf{F}$ is a constant force, then the work done by $\mathbf{F}$ in moving the particle along the path is the dot product $\mathbf{F} \cdot \mathbf{d}$:

$$\mathbf{F} \cdot \mathbf{d} = (\text{magnitude of force}) \times (\text{displacement in direction of force}).$$

If the path is curved, we can imagine that it is made up of a succession of infinitesimal straight-line displacements or that it is *approximated* by a finite number of straight-line displacements. Then (as in our derivation of the formulas for the path integral in the preceding section) we are led to the following formula for the work done by the force

[2] We thank Tanya Leise for suggesting this exercise.