Case 2. If $R \le \rho_1$ [that is, if (x_1, y_1, z_1) is inside the hole], then $|\rho - R| = \rho - R$ for ρ in $[\rho_1, \rho_2]$, and so

$$-V(0,0,R) = (Gm)\frac{2\pi}{R}\int_{\rho_1}^{\rho_2} \rho[\rho + R - (\rho - R)]\,d\rho = (Gm)4\pi\int_{\rho_1}^{\rho_2} \rho\,d\rho$$
$$= (Gm)2\pi\left(\rho_2^2 - \rho_1^2\right).$$

The result is independent of R, and so the potential V is *constant* inside the hole. Because the gravitational force is minus the gradient of V, we conclude that *there is no gravitational force inside a uniform hollow planet!*

We leave it to you to compute $V(0, 0, R)$ for the case $\rho_1 < R < \rho_2$.

A similar argument shows that the gravitational potential outside any *spherically symmetric* body of mass M (even if the density is variable) is $V = GMm/R$, where R is the distance to its center (which is its center of mass).

example 8 Find the gravitational potential acting on a unit mass of a spherical star with a mass $M = 3.02 \times 10^{30}$ kg at a distance of 2.25×10^{11} m from its center ($G = 6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$).

solution The negative potential is

$$-V = \frac{GM}{R} = \frac{6.67 \times 10^{-11} \times 3.02 \times 10^{30}}{2.25 \times 10^{11}} = 8.95 \times 10^8\ \mathrm{m^2/s^2}. \quad \blacktriangle$$

exercises

1. Find the coordinates of the center of mass of an isosceles triangle of uniform density bounded by the x axis, $y = ax$, and $y = -ax + 2a$.

2. Assuming uniform density, find the coordinates of the center of mass of the semicircle $y = \sqrt{r^2 - x^2}$, with $y \ge 0$.

3. Find the average of $f(x, y) = y \sin xy$ over $D = [0, \pi] \times [0, \pi]$.

4. Find the average of $f(x, y) = e^{x+y}$ over the triangle with vertices $(0, 0)$, $(0, 1)$, and $(1, 0)$.

5. Find the center of mass of the region between $y = x^2$ and $y = x$ if the density is $x + y$.

6. Find the center of mass of the region between $y = 0$ and $y = x^2$, where $0 \le x \le \frac{1}{2}$.

7. A sculptured gold plate D is defined by $0 \le x \le 2\pi$ and $0 \le y \le \pi$ (centimeters) and has mass density $\delta(x, y) = y^2 \sin^2 4x + 2$ (grams per square centimeter). If gold sells for \$7 per gram, how much is the gold in the plate worth?

8. In Exercise 7, what is the average mass density in grams per square centimeter?

9. (a) Find the mass of the box $[0, \frac{1}{2}] \times [0, 1] \times [0, 2]$, assuming the density to be uniform.

(b) Same as part (a), but with a mass density $\delta(x, y, z) = x^2 + 3y^2 + z + 1$.

10. Find the mass of the solid bounded by the cylinder $x^2 + y^2 = 2x$ and the cone $z^2 = x^2 + y^2$ if the density is $\delta = \sqrt{x^2 + y^2}$.

11. Find the mass of the solid ball of radius 5 with density given by

$$\delta(x, y, z) = 2x^2 + 2y^2 + 2z^2 + 1$$

assuming the center of the ball is at the origin.

12. A solid disk of radius 9 and height 2 is placed at the origin, so that it can be expressed by $x^2 + y^2 = 81$ and $0 \le z \le 2$. If the disk has a density given by

$$\delta(x, y, z) = 2x^2 + 2y^2 + 2z^2 + 1,$$

find its mass.

13. Find the center of mass of the region bounded by $x + y + z = 2$, $x = 0$, $y = 0$, and $z = 0$, assuming the density to be uniform.

14. Find the center of mass of the cylinder $x^2 + y^2 \le 1$, $1 \le z \le 2$ if the density is $\delta = (x^2 + y^2)z^2$.

15. Find the average value of $\sin^2 \pi z \cos^2 \pi x$ over the cube $[0, 2] \times [0, 4] \times [0, 6]$.

16. Find the average value of e^{-z} over the ball $x^2 + y^2 + z^2 \le 1$.

17. A solid with constant density is bounded above by the plane $z = a$ and below by the cone described in spherical coordinates by $\phi = k$, where k is a constant $0 < k < \pi/2$. Set up an integral for its moment of inertia about the z axis.

18. Find the moment of inertia around the y axis for the ball $x^2 + y^2 + z^2 \le R^2$ if the mass density is a constant δ.

19. Find the gravitational potential on a mass m of a spherical planet with mass $M = 3 \times 10^{26}$ kg, at a distance of 2×10^8 m from its center.

20. Find the gravitational force exerted on a 70-kg object at the position in Exercise 19.

21. A body W in xyz coordinates is called *symmetric with respect to a given plane* if for every particle on one side of the plane there is a particle of equal mass located at its mirror image through the plane.

(a) Discuss the planes of symmetry for an automobile shell.

(b) Let the plane of symmetry be the xy plane, and denote by W^+ and W^- the portions of W above and below the plane, respectively. By our assumption, the mass density $\delta(x, y, z)$ satisfies $\delta(x, y, -z) = \delta(x, y, z)$. Justify the following steps:

$$\begin{aligned}\bar{z} \cdot \iiint_W \delta(x, y, z)\,dx\,dy\,dz &= \iiint_W z\delta(x, y, z)\,dx\,dy\,dz \\ &= \iiint_{W^+} z\delta(x, y, z)\,dx\,dy\,dz + \iiint_{W^-} z\delta(x, y, z)\,dx\,dy\,dz \\ &= \iiint_{W^+} z\delta(x, y, z)\,dx\,dy\,dz + \iiint_{W^+} -w\delta(u, v, -w)\,du\,dv\,dw \\ &= 0.\end{aligned}$$

(c) Explain why part (b) proves that if a body is symmetrical with respect to a plane, then its center of mass lies in that plane.

(d) Derive this law of mechanics: *If a body is symmetric with respect to two planes, then its center of mass lies on their line of intersection.*

22. A uniform rectangular steel plate of sides a and b rotates about its center of mass with constant angular velocity ω.

(a) The kinetic energy equals $\frac{1}{2}$(mass)(velocity)^2. Argue that the kinetic energy of any element of mass $\delta\,dx\,dy$ (δ = constant) is given by $\delta(\omega^2/2)(x^2 + y^2)\,dx\,dy$, provided the origin $(0, 0)$ is placed at the center of mass of the plate.

(b) Justify the formula for kinetic energy:

$$\text{K.E.} = \iint_{\text{plate}} \delta\frac{\omega^2}{2}(x^2 + y^2)\,dx\,dy.$$

(c) Evaluate the integral, assuming that the plate is described by the inequalities $-a/2 \le x \le a/2$, $-b/2 \le y \le b/2$.

23. As is well known, the density of a typical planet is not constant throughout the planet. Assume that planet C.M.W. has a radius of 5×10^8 cm and a mass density (in grams per cubic centimeter)

$$\rho(x, y, z) = \begin{cases} \dfrac{3 \times 10^4}{r}, & r \ge 10^4 \text{ cm}, \\ 3, & r \le 10^4 \text{ cm}, \end{cases}$$

where $r = \sqrt{x^2 + y^2 + z^2}$. Find a formula for the gravitational potential outside C.M.W.

24. Let D be a region in the part of the xy plane with $x > 0$. Assume D has uniform density, area $A(D)$, and center of mass $(\bar{x}, \bar{y})$. Let W be the solid obtained by rotating D about the y axis. Show that the volume of W is given by

$$\text{vol}(W) = 2\pi\bar{x}A(D).$$

25. Use the previous exercise to show that if a doughnut is obtained by rotating the circle $(x - a)^2 + y^2 = r^2$ about the y axis, then the volume of the doughnut is $2\pi^2 ar^2$.