However, even if T is one-to-one near every point, and also onto, T need not be *globally* one-to-one. Thus, we must exercise caution (see Exercise 17).

Surprisingly, if D^* and D are elementary regions and $T\colon D^* \to D$ has the property that the determinant of $\mathbf{D}T(u, v)$ is not zero for any (u, v) in D^*, and if T maps the boundary of D^* in a one-to-one and onto manner to the boundary of D, then T is one-to-one and onto from D^* to D. (This proof is beyond the scope of this text.)

In summary, we have:

One-to-One and Onto Mappings A mapping $T\colon D^* \to D$ is ***one-to-one*** when it maps distinct points to distinct points. It is ***onto*** when the image of D^* under T is all of D.

A *linear* transformation of $\mathbb{R}^n$ to $\mathbb{R}^n$ given by multiplication by a matrix A is one-to-one and onto when and only when $\det A \neq 0$.

exercises

1. Determine if the following functions $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ are one-to-one and/or onto.

(a) $T(x, y) = (2x, y)$

(b) $T(x, y) = (x^2, y)$

(c) $T(x, y) = (\sqrt[3]{x}, \sqrt[3]{y})$

(d) $T(x, y) = (\sin x, \cos y)$

2. Determine if the following functions $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ are one-to-one and/or onto.

(a) $T(x, y, z) = (2x + y + 3z, 3y - 4z, 5x)$

(b) $T(x, y, z) = (y \sin x, z \cos y, xy)$

(c) $T(x, y, z) = (xy, yz, xz)$

(d) $T(x, y, z) = (e^x, e^y, e^z)$

3. Let D be a square with vertices $(0, 0), (1, 1), (2, 0), (1, -1)$ and D^* be a parallelogram with vertices $(0, 0), (1, 2), (2, 1), (1, -1)$. Find a linear map T taking D^* onto D.

4. Let D be a parallelogram with vertices $(0, 0), (-1, 3), (-2, 0), (-1, -3)$. Let $D^* = [0, 1] \times [0, 1]$. Find a linear map T such that $T(D^*) = D$.

5. Let $S^* = (0, 1] \times [0, 2\pi)$ and define $T(r, \theta) = (r\cos\theta, r\sin\theta)$. Determine the image set S. Show that T is one-to-one on S^*.

6. Define

$$T(x^*, y^*) = \left(\frac{x^* - y^*}{\sqrt{2}}, \frac{x^* + y^*}{\sqrt{2}}\right).$$

Show that T rotates the unit square, $D^* = [0, 1] \times [0, 1]$.

7. Let $D^* = [0, 1] \times [0, 1]$ and define T on D^* by $T(u, v) = (-u^2 + 4u, v)$. Find the image D. Is T one-to-one?

8. Let D^* be the parallelogram bounded by the lines $y = 3x - 4$, $y = 3x$, $y = \frac{1}{2}x$, and $y = \frac{1}{2}(x + 4)$. Let $D = [0, 1] \times [0, 1]$. Find a T such that D is the image of D^* under T.

9. Let $D^* = [0, 1] \times [0, 1]$ and define T on D^* by $T(x^*, y^*) = (x^* y^*, x^*)$. Determine the image set D. Is T one-to-one? If not, can we eliminate some subset of D^* so that on the remainder T is one-to-one?

10. Let D^* be the parallelogram with vertices at $(-1, 3), (0, 0), (2, -1)$, and $(1, 2)$, and D be the rectangle $D = [0, 1] \times [0, 1]$. Find a T such that D is the image set of D^* under T.

11. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the spherical coordinate mapping defined by $(\rho, \phi, \theta) \mapsto (x, y, z)$, where

$$x = \rho \sin\phi \cos\theta, \qquad y = \rho \sin\phi \sin\theta, \qquad z = \rho \cos\phi.$$

Let D^* be the set of points (ρ, ϕ, θ) such that $\phi \in [0, \pi]$, $\theta \in [0, 2\pi]$, $\rho \in [0, 1]$. Find $D = T(D^*)$. Is T one-to-one? If not, can we eliminate some subset of D^* so that, on the remainder, T will be one-to-one?

In Exercises 12 and 13, let $T(\mathbf{x}) = A\mathbf{x}$, where A is a 2×2 matrix.

12. Show that T is one-to-one if and only if the determinant of A is not zero.

13. Show that $\det A \neq 0$ if and only if T is onto.

14. Suppose $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is linear and is given by $T(\mathbf{x}) = A\mathbf{x}$, where A is a 2×2 matrix. Show that if $\det A \neq 0$, then T takes parallelograms onto parallelograms. [HINT: The general parallelogram in $\mathbb{R}^2$ can be described by the set of points $\mathbf{q} = \mathbf{p} + \lambda\mathbf{v} + \mu\mathbf{w}$ for $\lambda, \mu \in (0, 1)$ where $\mathbf{p}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^2$ with $\mathbf{v}$ not a scalar multiple of $\mathbf{w}$.]

15. A map $T : \mathbb{R}^2 \to \mathbb{R}^2$ is called **affine** if $T(\mathbf{x}) = A\mathbf{x} + \mathbf{v}$, where A is a 2×2 matrix, and $\mathbf{v}$ is a fixed vector in $\mathbb{R}^2$. Show that Exercises 12, 13, and 14 hold for T.

16. Suppose $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is as in Exercise 14 and that $T(P^*) = P$ is a parallelogram. Show that P^* is a parallelogram.

17. Show that T is not one-to-one.

University of South Carolina Columbia, S.C. 29208

6.2 The Change of Variables Theorem

Given two regions D and D^* in $\mathbb{R}^2$, a differentiable map T on D^* with image D—that is, $T(D^*) = D$—and any real-valued integrable function $f\colon D \to \mathbb{R}$, we would like to express $\iint_D f(x, y)\, dA$ as an integral over D^* of the composite function $f \circ T$. In this section we shall see how to do this.

Assume that D^* is a region in the uv plane and that D is a region in the xy plane. The map T is given by two coordinate functions:

$$T(u, v) = (x(u, v), y(u, v)) \qquad \text{for} \qquad (u, v) \in D^*.$$

At first, we might conjecture that

$$\iint_D f(x, y)\, dx\, dy \stackrel{?}{=} \iint_{D^*} f(x(u, v), y(u, v))\, du\, dv, \tag{1}$$

where $f \circ T(u, v) = f(x(u, v), y(u, v))$ is the composite function defined on D^*. However, if we consider the function $f\colon D \to \mathbb{R}^2$ where $f(x, y) = 1$, then equation (1) would imply

$$A(D) = \iint_D dx\, dy \stackrel{?}{=} \iint_{D^*} du\, dv = A(D^*). \tag{2}$$

But equation (2) will hold for only a few special cases and not for a general map T. For example, define T by $T(u, v) = (-u^2 + 4u, v)$. Restrict T to the unit square; that is, to the region $D^* = [0, 1] \times [0, 1]$ in the uv plane (see Figure 6.2.1). Then, as in

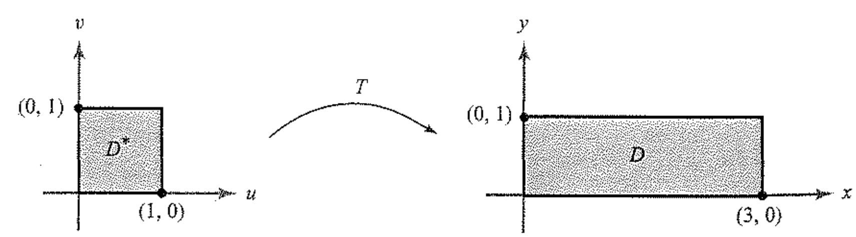

figure **6.2.1** The map $T\colon (u, v) \mapsto (-u^2 + 4u, v)$ takes the square D^* onto the rectangle D.