Because a continuous function on D takes on every value between its maximum and minimum values (this is the two-variable *intermediate-value theorem* proved in advanced calculus; see also Review Exercise 32), and because the number $[1/A(D)]\iint_D f(x, y)\,dA$ is, by inequality (6), between these values, there must be a point $(x_0, y_0) \in D$ with

$$f(x_0, y_0) = \frac{1}{A(D)}\iint_D f(x, y)\,dA,$$

which is precisely the conclusion of Theorem 5. ■

exercises

1. Change the order of integration, but do not evaluate, the following integrals:

(a) $\displaystyle\int_0^8\int_{1/2y}^4 dx\,dy$

(b) $\displaystyle\int_0^9\int_0^{\sqrt{y}} dx\,dy$

(c) $\displaystyle\int_0^4\int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} dx\,dy$

(d) $\displaystyle\int_{\pi/2}^{\pi}\int_0^{\sin x} dy\,dx$

2. Change the order of integration and evaluate:

$$\int_0^1\int_y^1 \sin(x^2)\,dx\,dy.$$

3. In the following integrals, change the order of integration, sketch the corresponding regions, and evaluate the integral both ways.

(a) $\displaystyle\int_0^1\int_x^1 xy\,dy\,dx$

(b) $\displaystyle\int_0^{\pi/2}\int_0^{\cos\theta} \cos\theta\,dr\,d\theta$

(c) $\displaystyle\int_0^1\int_1^{2-y} (x+y)^2\,dx\,dy$

(d) $\displaystyle\int_a^b\int_a^y f(x, y)\,dx\,dy$ (express your answer in terms of antiderivatives).

4. Find

(a) $\displaystyle\int_{-1}^1\int_{|y|}^1 (x+y)^2\,dx\,dy$

(b) $\displaystyle\int_{-3}^1\int_{-\sqrt{(9-y^2)}}^{\sqrt{(9-y^2)}} x^2\,dx\,dy$

(c) $\displaystyle\int_0^4\int_{y/2}^2 e^{x^2}\,dx\,dy$

(d) $\displaystyle\int_0^1\int_{\tan^{-1} y}^{\pi/4} (\sec^5 x)\,dx\,dy$

5. Change the order of integration and evaluate:

$$\int_0^1\int_{\sqrt{y}}^1 e^{x^3}\,dx\,dy.$$

6. Consider the intuitive fact that if a region D in $\mathbb{R}^2$ can be split into a disjoint union of subsets $D = D_1 \cup D_2$, then a double integral over D may also be divided into a sum of two integrals:

$$\iint_D f(x, y)\,dA = \iint_{D_1} f(x, y)\,dA + \iint_{D_2} f(x, y)\,dA.$$

(See Section 5.2 for the analogous statement over a rectangular box.) Are the following attempts to change the order of integration true or false?

(a)
$$\int_0^{\pi/4}\int_{\sin x}^{\cos x} dy\,dx = \int_0^{\sqrt{2}/2}\int_0^{\arcsin y} dx\,dy + \int_{\sqrt{2}/2}^2\int_0^{\arccos y} dx\,dy$$

(b)
$$\int_{-2}^2\int_0^{4-x^2} dy\,dx = \int_0^4\int_{-\sqrt{4-y}}^{\sqrt{4-y}} dx\,dy$$

(c)
$$\int_0^2\int_0^{(1/2)x} dy\,dx + \int_2^5\int_{(1/3)x-(2/3)}^1 dy\,dx = \int_0^1\int_{2y}^{3y+2} dx\,dy$$

(d)
$$\int_0^1\int_1^{e^x} dy\,dx = \int_1^e\int_{\ln y}^1 dx\,dy$$

7. If $f(x, y) = e^{\sin(x+y)}$ and $D = [-\pi, \pi] \times [-\pi, \pi]$, show that

$$\frac{1}{e} \le \frac{1}{4\pi^2}\iint_D f(x, y)\,dA \le e.$$

8. Show that

$$\frac{1}{2}(1 - \cos 1) \le \iint_{[0,1]\times[0,1]} \frac{\sin x}{1 + (xy)^4}\,dx\,dy \le 1.$$

9. If $D = [-1, 1] \times [-1, 2]$, show that

$$1 \leq \iint_D \frac{dx\,dy}{x^2 + y^2 + 1} \leq 6.$$

10. Using the mean-value inequality, show that

$$\frac{1}{6} \leq \iint_D \frac{dA}{y - x + 3} \leq \frac{1}{4},$$

where D is the triangle with vertices $(0, 0)$, $(1, 1)$, and $(1, 0)$.

11. Compute the volume of an ellipsoid with semiaxes a, b, and c. (HINT: Use symmetry and first find the volume of one half of the ellipsoid.)

12. Compute $\iint_D f(x, y)\, dA$, where $f(x, y) = y^2\sqrt{x}$ and D is the set of (x, y), where $x > 0$, $y > x^2$, and $y < 10 - x^2$.

13. Find the volume of the region determined by $x^2 + y^2 + z^2 \leq 10$, $z \geq 2$. Use the disk method from one-variable calculus and state how the method is related to Cavalieri's principle.

14. Evaluate $\iint_D e^{x-y}\, dx\, dy$, where D is the interior of the triangle with vertices $(0, 0)$, $(1, 3)$, and $(2, 2)$.

15. Evaluate $\iint_D y^3(x^2 + y^2)^{-3/2}\, dx\, dy$, where D is the region determined by the conditions $\frac{1}{2} \leq y \leq 1$ and $x^2 + y^2 \leq 1$.

16. Given that the double integral $\iint_D f(x, y)\, dx\, dy$ of a positive continuous function f equals the iterated integral $\int_0^1 \left[\int_{x^2}^{x} f(x, y)\, dy\right] dx$, sketch the region D and interchange the order of integration.

17. Given that the double integral $\iint_D f(x, y)\, dx\, dy$ of a positive continuous function f equals the iterated integral $\int_0^1 \left[\int_y^{\sqrt{2-y^2}} f(x, y)\, dx\right] dy$, sketch the region D and interchange the order of integration.

18. Prove that

$$2\int_a^b \int_x^b f(x) f(y)\, dy\, dx = \left(\int_a^b f(x)\, dx\right)^2.$$

[HINT: Notice that

$$\left(\int_a^b f(x)\, dx\right)^2 = \iint_{[a,b]\times[a,b]} f(x) f(y)\, dx\, dy.]$$

19. Show that (see Section 2.5, Exercise 29)

$$\frac{d}{dx}\int_a^x \int_c^d f(x, y, z)\, dz\, dy = \int_c^d f(x, y, z)\, dz + \int_a^x \int_c^d f_x(x, y, z)\, dz\, dy.$$

5.5 The Triple Integral

Triple integrals are needed for many physical problems. For example, if the temperature inside an oven is not uniform, determining the average temperature involves "summing" the values of the temperature function at all points in the solid region enclosed by the oven walls and then dividing the answer by the total volume of the oven. Such a sum is expressed mathematically as a triple integral.

Definition of the Triple Integral

Our objective now is to define the triple integral of a function $f(x, y, z)$ over a box (rectangular parallelepiped) $B = [a, b] \times [c, d] \times [p, q]$. Proceeding as in double integrals, we partition the three sides of B into n equal parts and form the sum

$$S_n = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} f(\mathbf{c}_{ijk})\, \Delta V,$$

where $\mathbf{c}_{ijk}$ is a point in B_{ijk}, the ijkth rectangular parallelepiped (or box) in the partition of B, and ΔV is the volume of B_{ijk} (see Figure 5.5.1).