To find the area of $D$, we substitute $f = 1$ in formula (2); this yields

$$\iint_D dA = \int_c^d (\psi_2(y) - \psi_1(y))\, dy.$$

Again, this result for area agrees with the results of single-variable calculus for the area of a region between two curves.

Either the method for $y$-simple or the method for $x$-simple regions can be used for integrals over simple regions.

It follows from formulas (1) and (2) that $\iint_D f\, dA$ is independent of the choice of the rectangle $R$ enclosing $D$ used in the definition of $\iint_D f\, dA$, because, if we had picked another rectangle enclosing $D$, we would have arrived at the same formula (1).

## exercises

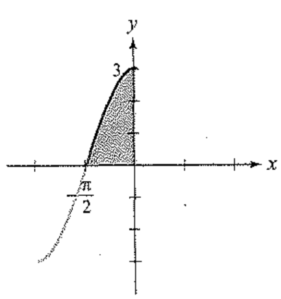
**1.** In parts (a) through (d) below, each iterated integral is an integral over a region $D$. Match the integral with the correct region of integration.

(a) $\displaystyle\int_1^2 \int_{\ln x}^{e^x} dy\, dx$

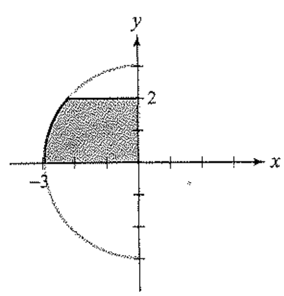
(b) $\displaystyle\int_0^2 \int_{(1/8)x}^{x^{1/3}} dy\, dx$

(c) $\displaystyle\int_0^2 \int_{-\sqrt{9-y^2}}^{0} dx\, dy$

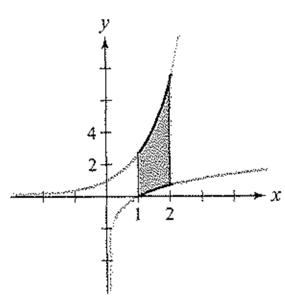
(d) $\displaystyle\int_0^3 \int_{\arccos y/3}^{0} dx\, dy$



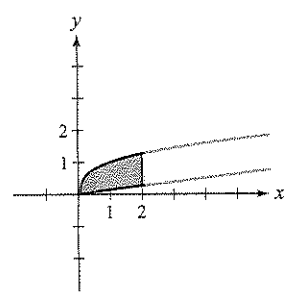

(i)




(ii)




(iii)




(iv)

**2.** Sketch the region $D$ in $\mathbb{R}^2$ that represents the region of integration:

(a) $\displaystyle\int_{-2}^2 \int_0^{4-y^2} (4 - x)\, dx\, dy$

(b) $\displaystyle\int_0^3 \int_{-x}^{x} (6 + y - 2x)\, dy\, dx$

**3.** Evaluate the following iterated integrals and draw the regions $D$ determined by the limits. State whether the regions are $x$-simple, $y$-simple, or simple.

(a) $\displaystyle\int_0^1 \int_0^{x^2} dy\, dx$

(b) $\displaystyle\int_1^2 \int_{2x}^{3x+1} dy\, dx$

(c) $\displaystyle\int_0^1 \int_1^{e^x} (x + y)\, dy\, dx$

(d) $\displaystyle\int_0^1 \int_{x^3}^{x^2} y\, dy\, dx$

**4.** Evaluate the following integrals and sketch the corresponding regions.

(a) $\displaystyle\int_{-3}^2 \int_0^{y^2} (x^2 + y)\, dx\, dy$

(b) $\displaystyle\int_{-1}^1 \int_{-2|x|}^{|x|} e^{x+y}\, dy\, dx$

(c) $\displaystyle\int_0^1 \int_0^{(1-x^2)^{1/2}} dy\, dx$

(d) $\displaystyle\int_0^{\pi/2} \int_0^{\cos x} y \sin x\, dy\, dx$

(e) $\displaystyle\int_0^1 \int_{y^2}^{y} (x^n + y^m)\, dx\, dy, \qquad m, n > 0$

(f) $\displaystyle\int_{-1}^0 \int_0^{2(1-x^2)^{1/2}} x\, dy\, dx$

5. Use double integrals to compute the area of a circle of radius $r$.

6. Using double integrals, determine the area of an ellipse with semiaxes of length $a$ and $b$.

7. What is the volume of a barn that has a rectangular base 20 ft by 40 ft, vertical walls 30 ft high at the front (which we assume is on the 20-ft side of the barn), and 40 ft high at the rear? The barn has a flat roof. Use double integrals to compute the volume.

8. Let $D$ be the region bounded by the positive $x$ and $y$ axes and the line $3x + 4y = 10$. Compute

$$\iint_D (x^2 + y^2)\, dA.$$

9. Let $D$ be the region bounded by the $y$ axis and the parabola $x = -4y^2 + 3$. Compute

$$\iint_D x^3 y\, dx\, dy.$$

10. Evaluate $\int_0^1 \int_0^{x^2} (x^2 + xy - y^2)\, dy\, dx$. Describe this iterated integral as an integral over a certain region $D$ in the $xy$ plane.

11. Let $D$ be the region given as the set of $(x, y)$, where $1 \le x^2 + y^2 \le 2$ and $y \ge 0$. Is $D$ an elementary region? Evaluate $\iint_D f(x, y)\, dA$, where $f(x, y) = 1 + xy$.

12. Evaluate the following double integral:

$$\iint_D \cos y\, dx\, dy,$$

where the region $D$ is bounded by $y = 2x$, $y = x$, $x = \pi$, and $x = 2\pi$.

13. Evaluate the following double integral:

$$\iint_D xy\, dA,$$

where the region $D$ is the triangular region whose vertices are $(0, 0)$, $(0, 2)$, $(2, 0)$.

14. Use the formula $A(D) = \iint_D dx\, dy$ to find the area enclosed by one period of the sine function $\sin x$, for $0 \le x \le 2\pi$, and the $x$ axis.

15. Find the volume of the region inside the surface $z = x^2 + y^2$ and between $z = 0$ and $z = 10$.

16. Set up the integral required to calculate the volume of a cone of base radius $r$ and height $h$.

17. Evaluate $\iint_D y\, dA$, where $D$ is the set of points $(x, y)$ such that $0 \le 2x/\pi \le y$, $y \le \sin x$.

18. From Exercise 9, Section 5.2,

$$\int_a^b \int_c^d f(x) g(y)\, dy\, dx = \left(\int_a^b f(x)\, dx\right) \left(\int_c^d g(y)\, dy\right).$$

Is it true that $\iint_D f(x) g(y)\, dx\, dy = \left(\int_a^b f(x)\, dx\right) \left(\int_{\phi_1(a)}^{\phi_2(b)} g(y)\, dy\right)$ for $y$-simple regions?

19. Let $D$ be a region given as the set of $(x, y)$ with $-\phi(x) \le y \le \phi(x)$ and $a \le x \le b$, where $\phi$ is a nonnegative continuous function on the interval $[a, b]$. Let $f(x, y)$ be a function on $D$ such that $f(x, y) = -f(x, -y)$ for all $(x, y) \in D$. Argue that $\iint_D f(x, y)\, dA = 0$.

20. Use the methods of this section to show that the area of the parallelogram $D$ determined by two planar vectors $\mathbf{a}$ and $\mathbf{b}$ is $|a_1 b_2 - a_2 b_1|$, where $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$.

21. Describe the area $A(D)$ of a region as a limit of areas of inscribed rectangles, as in Example 3.

## 5.4 Changing the Order of Integration

Suppose that $D$ is a simple region—that is, it is both $x$-simple and $y$-simple. Thus, it can be given as the set of points $(x, y)$ such that

$$a \le x \le b, \qquad \phi_1(x) \le y \le \phi_2(x),$$

and also as the set of points $(x, y)$ such that

$$c \le y \le d, \qquad \psi_1(y) \le x \le \psi_2(y).$$