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Quiz for April 4, 2005

1. Let $\zeta = e^{\frac{2\pi i}{17}}$ and let K be the field $\mathbb{Q}[\zeta]$. Find an element u_2 in K with $\dim_{\mathbb{Q}} \mathbb{Q}[u_2] = 4$.

ANSWER: I use Galois Theory. If $\sigma \in \text{Aut}_{\mathbb{Q}} K$, then $\sigma(\zeta)$ must be a root of the minimal polynomial of ζ. So $\sigma(\zeta)$ must equal ζ^j for some j with $1 \le j \le 16$. Once we know what σ does to ζ, then we know what σ does to every element of K. Also, for each j, with $1 \le j \le 16$, we know that ζ and ζ^j have the same minimal polynomial over $\mathbb{Q}$ and therefore there is a ring isomorphism $\sigma_j \colon \mathbb{Q}[\zeta] \to \mathbb{Q}[\zeta^j]$, with $\sigma_j(\zeta) = \zeta^j$. Of course, we know that $\mathbb{Q}[\zeta^j] = K$. Put all of the above information together to see that $\text{Aut}_{\mathbb{Q}} K = \{\sigma_j \mid 1 \le j \le 16\}$. It turns out that $\text{Aut}_{\mathbb{Q}} K$ is a cyclic group generated by σ_3. Henceforth, I will write σ in place of σ_3. At any rate, the subgroups of $\text{Aut}_{\mathbb{Q}} K =< \sigma >$. are:

$$< 1 > \subseteq < \sigma^8 > \subseteq < \sigma^4 > \subseteq < \sigma^2 > \subseteq < \sigma > = \text{Aut}_{\mathbb{Q}} K.$$

The Fundamental Theorem of Galois Theory tells us that the subfields of K which lie between K and $\mathbb{Q}$ are

$$K = K^{<1>} \supseteq K^{<\sigma^8>} \supseteq K^{<\sigma^4>} \supseteq K^{<\sigma^2>} \supseteq K^{<\sigma>} = \mathbb{Q}.$$

Furthermore, for each intermediate field K^H, we have $\dim_{K^H} K = |H|$. So, $\dim_{K^{<\sigma^4>}} K = 4$ and therefore, $\dim_{\mathbb{Q}} K^{<\sigma^4>}$ is also 4 because $\dim_{\mathbb{Q}} K = 16$ and $\dim_{\mathbb{Q}} K = \dim_{\mathbb{Q}} K^{<\sigma^4>} \dim_{K^{<\sigma^4>}} K$.

Our job is to find $u_2 \in K^{<\sigma^4>}$, with $u_2 \notin K^{<\sigma^2>}$. This is easy. We notice that

$$\sigma^4(\zeta) = \zeta^{13}, \quad \sigma^4(\zeta^{13}) = \zeta^{16}, \quad \sigma^4(\zeta^{16}) = \zeta^4, \quad \text{and} \quad \sigma^4(\zeta^4) = \zeta.$$

We take

$$u_2 = \zeta + \zeta^{13} + \zeta^{16} + \zeta^4.$$

We just calculated that $\sigma^4(u_2) = u_2$. **Make sure you understand that claim!** So, $u_2 \in \dim_{K^{<\sigma^4>}}$. It is clear that $u_2 \notin K^{<\sigma^2>}$ because,

$$\sigma^2(u_2) = \zeta^9 + \zeta^{15} + \zeta^8 + \zeta^2.$$

We know that $\sigma^2(u_2) \ne u_2$ because we know that $\{\zeta^j \mid 1 \le j \le 16\}$ is linearly independent.

If you wanted to find the minimal polynomial of u_2 over $\mathbb{Q}$, then you must do some linear algebra and find a relation on $1, u_2, u_2^2, u_2^3, u_2^4$. The point is that I listed 5 elements from the vector space $\mathbb{Q}[u_2]$, but I believe that $\dim_{\mathbb{Q}} \mathbb{Q}[u_2] = 4$; so the 5 elements that I listed MUST be linearly dependent. The relation on the 5 elements gives a polynomial of degree 4 that u_2 satisfies.