PRINT Your Name:____________________________________

## Quiz for February 11, 2005

**Definition.** Let $I$ be an ideal of the ring $R$, with $I \neq R$. The ideal $I$ is a *prime ideal* of $R$ if, whenever $a$ and $b$ are in $R$ with $ab \in I$, then $a \in I$ or $b \in I$.

**Definition.** Let $I$ be an ideal of the ring $R$, with $I \neq R$. The ideal $I$ is a *maximal ideal* of $R$ if $R$ is the only ideal of $R$ which properly contains $I$.

**Definition.** The domain $R$ is a *Principal Ideal Domain* if every ideal in $R$ is principal.

1. Prove that every non-zero prime ideal in a Principal Ideal Domain is a maximal ideal.

**ANSWER:** Let $I$ be a non-zero prime ideal of the Principal Ideal Domain $R$. We know that $I = (r)$ for some element $r$ of $R$. Let $J$ be an ideal of $R$ with $I \subseteq J \subseteq R$. The ring $R$ is a PID, so $J = (s)$ for some element $s$ of $R$. We have $r \in I \subseteq J = (s)$; so, $r = st$ for some element $t$ in $R$. The product $st$ is in the prime ideal $I$. It follows that either $s \in I$ or $t \in I$.
**Case 1.** If $s \in I$, then $s = ar$ for some element $a$ in $R$ and $r = st = art$. The ring $R$ is a domain; hence, $1 = at$. In other words, $a$ is a unit and $I = J$.
**Case 2.** If $t \in I$, then $t = rb$ for some $b \in R$ and $r = st = srb$. The ring $R$ is a domain; hence, $1 = sb$. In this case $s$ is a unit and $J = R$.

We have shown that there do not exist any ideals $J$ of $R$ with $I \subsetneq J \subsetneq R$; and therefore, $I$ is a maximal ideal of $R$.