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Quiz for January 21, 2005

Prove that the composition of two ring homomorphisms is a ring homomorphism.

ANSWER: Suppose $\phi\colon R \to S$ and $\theta\colon S \to T$ are ring homomorphisms.

- We see that

$$(\theta \circ \phi)(1) = \theta(\phi(1))$$

by the definition of composition;

$$\theta(\phi(1)) = \theta(1)$$

because ϕ is a ring homomorphism; and $\theta(1) = 1$ because θ is a ring homomorphism. Thus,

$$(\theta \circ \phi)(1) = 1.$$

- Take r_1 and r_2 from R. We see that

$$(\theta \circ \phi)(r_1 + r_2) = \theta(\phi(r_1 + r_2))$$

by the definition of composition;

$$\theta(\phi(r_1 + r_2)) = \theta(\phi(r_1) + \phi(r_2))$$

because ϕ is a ring homomorphism;

$$\theta(\phi(r_1) + \phi(r_2)) = \theta(\phi(r_1)) + \theta(\phi(r_2))$$

because θ is a ring homomorphism; and

$$\theta(\phi(r_1)) + \theta(\phi(r_2)) = (\theta \circ \phi)(r_1) + (\theta \circ \phi)(r_2)$$

by the definition of composition. Thus,

$$(\theta \circ \phi)(r_1 + r_2) = (\theta \circ \phi)(r_1) + (\theta \circ \phi)(r_2).$$

- Take r_1 and r_2 from R. We see that

$$(\theta \circ \phi)(r_1 r_2) = \theta(\phi(r_1 r_2))$$

by the definition of composition;

$$\theta(\phi(r_1 r_2)) = \theta(\phi(r_1)\phi(r_2))$$

because ϕ is a ring homomorphism;

$$\theta(\phi(r_1)\phi(r_2)) = \theta(\phi(r_1))\theta(\phi(r_2))$$

because θ is a ring homomorphism; and

$$\theta(\phi(r_1))\theta(\phi(r_2)) = (\theta \circ \phi)(r_1)(\theta \circ \phi)(r_2)$$

by the definition of composition. Thus,

$$(\theta \circ \phi)(r_1 r_2) = (\theta \circ \phi)(r_1)(\theta \circ \phi)(r_2).$$