Homework Problems Math 547 April 4, 2005

1. Let $K \subseteq L$ be fields, $f(x)$ be an irreducible polynomial of $K[x]$, and α_1 and α_2 be elements of L with $f(\alpha_1) = f(\alpha_2) = 0$. Prove that there exists a ring isomorphism $\sigma\colon K[\alpha_1] \to K[\alpha_2]$ with $\sigma(\alpha_1) = \alpha_2$ and $\sigma(k) = k$ for all $k \in K$.

2. Let K_1 and K_2 be subfields of the field L. Suppose that $\sigma\colon K_1 \to K_2$ is a ring isomorphism. Let $f_1(x) = a_0 + a_1x + \cdots + a_nx^n$ be an irreducible polynomial of $K_1[x]$. Let $f_2(x)$ be the polynomial $f_2(x) = \sigma(a_0) + \sigma(a_1)x + \cdots + \sigma(a_n)x^n$ in $K_2[x]$. Let α_1 and α_2 be elements of L with $f_1(\alpha_1) = f_2(\alpha_2) = 0$. Prove that there exists a ring isomorphism $\tau\colon K_1[\alpha_1] \to K_2[\alpha_2]$ with $\tau(\alpha_1) = \alpha_2$ and $\tau(k_1) = \sigma(k_1)$ for all $k_1 \in K_1$.

3. Let $K \subseteq \mathbb{C}$ be fields, and let $f(x)$ be an irreducible polynomial in $K[x]$. Prove that $f(x)$ has DISTINCT roots in $\mathbb{C}$.

4. Let K_1 and K_2 be fields with $\mathbb{Q} \subseteq K_1, K_2 \subseteq \mathbb{C}$. Suppose $\phi\colon K_1 \to K_2$ is a ring isomorphism. Let $f_1(x) = a_0 + a_1x + \cdots + a_nx^n$ be a polynomial of $K_1[x]$, and let $f_2(x) = \varphi(a_0) + \varphi(a_1)x + \cdots + \varphi(a_n)x^n$ be the corresponding polynomial of $K_2[x]$. Let L_1 be the splitting field of f_1 over K_1 and L_2 be the splitting field of f_2 over K_2. Let S be the set of ring homomorphisms

$$S = \{\Phi\colon L_1 \to L_2 \mid \Phi(k) = \phi(k) \text{ for all } k \in K_1\}.$$

Prove that the number of elements of S is equal to $\dim_{K_1} L_1$.

5. Let K be a subfield of $\mathbb{C}$, $f(x)$ be a polynomial in $K[x]$, and E be the splitting field of f over K. Prove that the number of elements of $\mathrm{Aut}_K E = \dim_K E$.