**Homework Problems  Math 547  February 11, 2005**

Suppose that the field $F$ is a subring of the ring $R$. (For example, the field $\mathbb{Q}$ is a subring of all of the following rings: $\mathbb{Q}[x]$, $\mathbb{R}$, $\mathbb{Q}[i]$, and $\frac{\mathbb{Q}[x]}{I}$ for some ideal $I$ of $\mathbb{Q}[x]$.) Notice that $R$ is automatically a vector space over $F$. (Recall from your Linear Algebra class that a vector space over the field $F$ is an abelian group $V$ which admits scalar multiplication by elements of $F$. The scalar multiplication has to satisfy a handful of properties. In our situation, $R$ is an abelain group and it is possible to multiply elements of $R$ by elements of $F$ (even more is possible for us). All of the rules about scalar multiplication in a vector space automatically hold in the ring $R$.)

1. (a) What is the dimension of the vector space $\mathbb{Q}[i]$ over the field $\mathbb{Q}$? (You probably should find a basis for $\mathbb{Q}[i]$ over $\mathbb{Q}$. In other words, you want a set of elements from $\mathbb{Q}[i]$ which span $\mathbb{Q}[i]$ over $\mathbb{Q}$ and are linearly independent over $\mathbb{Q}$. Of course, $\mathbb{Q}[i]$ is the smallest subring of $\mathbb{C}$ which contains $\mathbb{Q}$ and $i$.)
   (b) Let $f$ be the polynomial $a_1 + a_1 x + \cdots + a_{n-1}x^{n-1} + x^n$ in $\mathbb{Q}[x]$. What is the dimension of the vector space $\frac{\mathbb{Q}[x]}{(f)}$ over $\mathbb{Q}$?
   (c) Suppose that $E \subseteq F \subseteq K$ are fields and that $u_1, \dots, u_n$ is a basis of $F$ over $E$ and that $v_1, \dots, v_m$ is a basis of $K$ over $F$. Prove that $\{u_i v_j \mid 1 \le i \le n,\ 1 \le j \le m\}$ is a basis for $K$ over $E$.
   (d) Let $\mathbb{Q}(\sqrt{2}, i)$ be the smallest subfield of $\mathbb{C}$ which contains $\sqrt{2}$, $i$, and $\mathbb{Q}$. Find a basis for $\mathbb{Q}(\sqrt{2}, i)$ over $\mathbb{Q}$.

2. (a) Let $\alpha$ be a complex number. Suppose that the ring $\mathbb{Q}[\alpha]$ has finite dimension as a vector space over $\mathbb{Q}$. Prove that $\mathbb{Q}[\alpha]$ is a field. (As always, $\mathbb{Q}[\alpha]$ is the smallest ring which contains $\mathbb{Q}$ and $\alpha$.)
   (b) If $\alpha = e^{\frac{2\pi i}{23}}$, then what is the dimension of $\mathbb{Q}[\alpha]$ over $\mathbb{Q}$?
   (c) Give an example of a complex number $\alpha$ for which $\mathbb{Q}[\alpha]$ is an infinite dimensional vector space over $\mathbb{Q}$.
   (d) Let $E \subseteq F$ be fields. Suppose that the dimension of $F$ as a vector space over $E$ is a prime integer. Prove that if $u$ is any element of $F$ with $u \notin E$, then $F = E[u]$.
   (e) Prove that there aren't any rings $R$ with $\mathbb{Q} \subsetneq R \subsetneq \mathbb{Q}[\sqrt[7]{2}]$.