Math 547, Final Exam, Spring , 2005

The exam is worth 100 points. Each problem is worth 11 1/9 points.

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Take enough space for each problem. Turn in your solutions in the order: problem 1, problem 2, ... ; although, by using enough paper, you can do the problems in any order that suits you.

I will e-mail your grade to you as soon as I finish grading the exams.

I will post the solutions on my website later today.

1. Let $K \subseteq L$ be fields, $f(x)$ be a polynomial in $K[x]$, $\sigma \in \operatorname{Aut}_K L$, and $\ell \in L$. Suppose that $f(\ell) = 0$. Prove $f(\sigma(\ell)) = 0$. Give all details.

2. Let $K \subseteq L$ be fields, $f(x)$ be an irreducible polynomial of $K[x]$, and α_1 and α_2 be elements of L with $f(\alpha_1) = f(\alpha_2) = 0$. Prove that there exists a ring isomorphism $\sigma\colon K[\alpha_1] \to K[\alpha_2]$ with $\sigma(\alpha_1) = \alpha_2$ and $\sigma(k) = k$ for all $k \in K$. Give all details.

3. State the Fundamental Theorem of Galois Theory. Please give hypotheses and conclusions.

4. Let I be an ideal of the ring R. Prove that I is a maximal ideal of R if and only if R/I is a field.

5. Prove that $\mathbb{Q}[x]$ is a Principal Ideal Domain.

6. Let I be an ideal in a Principal Ideal Domain R. Prove that the following statements are equivalent. (That is, if one of the statements is true, then they all are true. If one of the statements is false, then they all are false.)
   (a) There is an irreducible element r of R with $I = (r)$.
   (b) The ideal I is a non-zero prime ideal.
   (c) The ideal I is a maximal ideal.

7. Let K be the splitting field of $x^5 - 2$ over $\mathbb{Q}$. We have shown that $K = \mathbb{Q}[\sqrt[5]{2}, \omega]$, where $\omega = e^{\frac{2\pi i}{5}}$. We have also shown that $\dim_{\mathbb{Q}} K = 20$, and that there exist auotmorphisms σ, τ in $\operatorname{Aut}_{\mathbb{Q}} K$ with

$$\sigma(\sqrt[5]{2}) = \sqrt[5]{2} \quad \sigma(\omega) = \omega^2$$

$$\tau(\sqrt[5]{2}) = \omega\sqrt[5]{2} \quad \tau(\omega) = \omega.$$

   Furthermore we have shown that $\operatorname{Aut}_{\mathbb{Q}} K$ is generated by σ and τ. You do not have to re-prove any of the above facts. However, I do want complete details for the following things: Find a field E with $\mathbb{Q} \subseteq E \subseteq K$ and $\dim_{\mathbb{Q}} E = 2$. Find the subgroup H of $\operatorname{Aut}_{\mathbb{Q}} K$ with $K^H = E$. ("Find" means tell me generators.)

8. Let H be the subgroup $<(1,2,3,4),(1,3)>$ of S_4. Let S_4/H be the set of left cosets of H in S_4. Let H act on S_4/H by left translation. In other words, if h is in H and gH is a left coset of H in S_4 (i.e., $g \in S_4$), then h sends gH to the left coset hgH.
   (a) Find the orbit of each element of S_4/H.
   (b) Find the normalizer of H in S_4. Recall that the normalizer of H in S_4 is
   $$N_{S_4}(H) = \{g \in S_4 \mid gHg^{-1} = H\}.$$

9. Let K be the splitting field of $x^{17} - 1$ over $\mathbb{Q}$. We know that $K = \mathbb{Q}[\omega]$, for $\omega = e^{\frac{2\pi i}{17}}$. We also know that $\mathrm{Aut}_{\mathbb{Q}} K$ is the cyclic group of order 16 which is generated by the automorphism σ where $\sigma(\omega) = \omega^3$. You do not have to re-prove any of the above facts. However, I do want complete details for the following things: Find a subgroup H of $\mathrm{Aut}_{\mathbb{Q}} K$ with 8 elements. Find the field K^H. ("Find" means tell me generators.)