Review sheet for Exam 3 – First Installment

Be able to do all of the assigned Homework problems, all of the problems on Exams 1 and 2, and all of the problems on the Review sheets for Exams 1 and 2.

1. Be able to define: prime ideal, maximal ideal, Principal Ideal Domain, irreducible element, Unique Factorization domain.

2. Let $f(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$ be a polynomial in $\mathbb{Z}[x]$. Suppose that a and b are relatively prime integers with $f(\frac{a}{b}) = 0$. Prove that a divides c_0 in $\mathbb{Z}$ and b divides c_n in $\mathbb{Z}$.

3. Write $\frac{1}{\sqrt[3]{2}}$, $\frac{1}{1+\sqrt[3]{2}}$, and $\frac{1}{1+2\sqrt[3]{2}+3(\sqrt[3]{2})^2}$ in the form $a + b\sqrt[3]{2} + c(\sqrt[3]{2})^2$ with a, b, c in $\mathbb{Q}$.

4. Let f be a polynomial in $\mathbb{Z}[x]$. Suppose that the coefficients of f are relatively prime. Prove that f is irreducible in $\mathbb{Z}[x]$ if and only if f is irreducible in $\mathbb{Q}[x]$.

5. Let R be a domain. Suppose that there exists a field F with $F \subseteq R$ and $\dim_F R < \infty$. Prove that R is a field.

6. Give an example of a ring R and a field F with $F \subseteq R$, $\dim_F R < \infty$, and R is not a field.

7. Let R be a domain in which every ideal is finitely generated. Let r be an element of R which is not zero and not a unit. Prove that r is equal to a finite product of irreducible elements.