Review sheet for Exam 2

You should be able to do the following problems when you take Exam 2. (I will make the review sheet for Exam 3 before I give you Exam 3.)

1. **Everything from Exam 1 and the review sheet for Exam 1.**

2. Let I be an ideal in a Principal Ideal Domain R. Prove that the following statements are equivalent. (That is, if one of the statements is true, then they all are true. If one of the statements is false, then they all are false.)
   (a) There is an irreducible element r of R with $I = (r)$.
   (b) The ideal I is a prime ideal.
   (c) The ideal I is a maximal ideal.

3. Give an example of a ring R and an element r of R with r irreducible in R, but (r) not a prime ideal. Explain.

4. Give an example of a ring R and an ideal I of R with I a prime ideal, but I not a maximal ideal. Explain.

5. Let R be the ring $\mathbb{R}[x, y, z]$ and let I be the set

$$\{f \in R \mid f(p) = 0 \text{ for all points } p \text{ on the coordinate axes in } 3\text{-space}\}.$$

(a) Prove that I is an ideal of R.
(b) Find a generating set for I. Prove that your answer is correct.

6. The ideal $I = (108, 72, 180)$ of $\mathbb{Z}$ must be a principal ideal. Find a generator for I. Prove that your answer is correct.

7. The ideal $I = (x^5+x^4-2x^3-2x^2+x+1, x^5-x^4-2x^3+2x^2+x-1, x^5-2x^3+x)$ of $\mathbb{Q}[x]$ must be a principal ideal. Find a generator for I. Prove that your answer is correct.

8. Let R be a ring. Suppose that every ideal of R is finitely generated. Let $I_0 \subseteq I_1 \subseteq \dots$ be a chain of ideals in R. Prove that there exists an integer n such that $I_n = I_{n+1} = \dots$.

9. Give an example of ring R and an ideal I so that I requires an infinite number of generators.

10. Prove that $\mathbb{Q}[x]$ is a Principal ideal domain.

11. Prove that $\mathbb{Z}$ is a Principal Ideal Domain.

12. Find the minimal polynomial of $\sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$.

13. Find the minimal polynomial of $e^{\frac{2\pi i}{9}}$ over $\mathbb{Q}$.

14. Find the minimal polynomial of $e^{\frac{2\pi i}{7}}$ over $\mathbb{Q}$.

15. Let I be an ideal in the ring R. Prove that I is a prime ideal if and only if $\frac{R}{I}$ is a domain.

16. Let I be an ideal in the ring R. Prove that I is a maximal ideal if and only if $\frac{R}{I}$ is a field.

17. Let $F \subseteq E$ be fields with E a finite dimensional vector space over F. Let R be a ring with $F \subseteq R \subseteq E$. Prove that R is a field.