PRINT Your Name:____________________________________

Quiz for February 4, 2010

The quiz is worth 5 points. **Remove EVERYTHING from your desk except this quiz and a pen or pencil.**

Let G be a group, and let $a \in G$. Define the *centralizer* of a to be the set of elements in G that commute with a; that is, $C(a) = \{x \in G \mid xa = ax\}$.

(a) Prove that $C(a)$ is a subgroup of G.

(b) List the elements in the centralizer of σ in D_4. Explain what you are doing.

ANSWER: We start with (a).

Closure: Take x and y in $C(a)$. We show that xy is in $C(a)$. We show that $(xy)a = a(xy)$. We use associativity and the fact that y is in $C(a)$ to see that

$$(xy)a = x(ya) = x(ay).$$

Now we use associativity (twice) and the fact that x is in $C(a)$ to see:

$$x(ay) = (xa)y = (ax)y = a(xy).$$

Thus, $(xy)a = a(xy)$, as required.

Associativity: The operation is associative on all of G; so it is also associative on the subset $C(a)$ of G.

Identity: The identity element of G commutes with evey element of G; hence, the idenity element of G commutes with a and is in $C(a)$.

Inverses: Let x be an element of $C(a)$. We know that x has an inverse in G; let us call that inverse y. We would like to show that y is in $C(a)$. Start with the statement $xa = ax$ (which holds because $x \in C(a)$. Multiply on the left by y and multiply on the right by y:

$$\boxed{y(xa)y = y(ax)y}.$$

Associate like crazy:

$$ay = (yx)ay = \boxed{y(xa)y = y(ax)y} = ya(xy) = ya,$$

to see that $ay = ya$; hence, $y \in C(a)$.

Now we do (b). We see that $C(\sigma) = \{\mathrm{id}, \sigma, \rho^2, \sigma\rho\}$. It is not hard to see that the listed elements commute with σ. Maybe the only interesting one is:

$$\sigma\rho^2 = \rho^3\sigma\rho = \rho^3\rho^3\sigma = \rho^2\sigma.$$

It is also not hard to see that σ does not commute with ρ or ρ^3 because $\rho\sigma = \sigma\rho^3$. Similarily one checks that σ does not commute with $\sigma\rho$ or $\sigma\rho^3$. $\square$