PRINT Your Name:__________________________________

Quiz for April 15, 2010

The quiz is worth 5 points. **Remove EVERYTHING from your desk except this quiz and a pen or pencil.**

Exhibit an isomorphism $\phi\colon U \to G$, where U is the unit circle group and G is a subgroup of $\mathrm{GL}_2(\mathbb{R})$. Tell me what G is. Tell me what ϕ is. Prove that ϕ is an isomorphism.

ANSWER: Define $\varphi\colon U \to G$ by $\varphi(a+b\imath) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. (The other way to think of this function is that the number $\cos\theta + \imath\sin\theta$ is being sent to the matrix that represents rotation by θ.)

φ **is a homomorphism:** Take $a+b\imath$ and $c+d\imath$ from U. We see that

$$\varphi((a+b\imath)(c+d\imath)) = \varphi(ac-bd+\imath(ad+bc)) = \begin{bmatrix} ac-bd & -(ad+bc) \\ ad+bc & ac-bd \end{bmatrix}.$$

On the other hand,

$$\varphi(a+b\imath)\varphi(c+d\imath) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac-bd & -ad-bc \\ bc+ad & -bd+ac \end{bmatrix}.$$

φ **is one-to-one:** We show that $\ker\varphi = \{1\}$. If $\varphi(a+b\imath) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $a=1$ and $b=0$; so, $a+\imath b = 1$.

We conclude that φ is an isomorphism from U to $\operatorname{im}\varphi$.

The image of φ is called the rotation group or $\mathrm{SO}_2(R)$. It consists of all orthogonal matrices of determinant one.