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Quiz for September 22, 2004

Let G be a finite group with an even number of elements. Prove that there must exist an element $a \in G$ with $a \neq \mathrm{id}$, but $a^2 = \mathrm{id}$.

ANSWER: Observe that G is the disjoint union of the sets

$$Y = \{g \in G \mid g^2 = \mathrm{id}\} \quad \text{and} \quad N = \{g \in G \mid g^2 \neq \mathrm{id}\}.$$

The set Y always contains at least one element, namely id. Observe that if $g \in N$, then g^{-1} is also in N and $g \neq g^{-1}$. It follows that N may be partitioned into a collection of subsets each of which consists of a pair of elements which are inverses of one another. Thus, N contains an even number of elements. The hypothesis ensures that the group G contains an even number of elements. We conclude that Y contains an even number of elements. Since Y contains at least one element, we now know that Y must contain at least two elements. In other words, there does exist an element g in G with $g \neq \mathrm{id}$, but $g^2 = \mathrm{id}$.