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Quiz for January 22, 2004

Let $S = \mathbb{R} \setminus \{-1\}$. Define $*$ on S by $a * b = a + b + ab$. Prove that $(S, *)$ is a group.

ANSWER:

Closure: Take a, b from S . We must show that $a*b$ is in S . Well, $a*b = a+b+ab$, which is clearly a real number. We must check that $a+b+ab$ is not equal to -1 . If $a+b+ab$ were equal to -1 , then $a+b+ab=-1$; so, $1+a+b+ab=0$; that is, $(1+a)(1+b)=0$; so $a=-1$ or $b=-1$. On the other hand, a and b are in S ; so neither a nor b is -1 . We conclude that $a+b+ab \neq -1$; therefore, $a+b+ab \in S$

Associativity: Take a , b , and c from S . Observe that

$$a*(b*c) = a*(b+c+bc) = a+(b+c+bc)+a(b+c+bc) = a+b+c+ab+ac+bc+abc.$$

On the other hand,

$$(a*b)*c = (a+b+ab)*c = (a+b+ab)+c+(a+b+ab)c = a+b+c+ab+ac+bc+abc.$$

We see that $a*(b*c) = (a*b)*c$.

Identity: The number 0 is the identity element of S because $a*0 = a+0+a(0) = a$ and $0*a = 0+a+0(a) = a$ for all $a \in S$.

Inverses: Take $a \in S$. The inverse of a is $\frac{-a}{1+a}$ because

$$a * \frac{-a}{1+a} = a + \frac{-a}{1+a} + a\frac{-a}{1+a} = a + \frac{-a(1+a)}{1+a} = a - a = 0.$$

The operation * is commutative; so, $\frac{-a}{1+a} * a$ is also equal to 0 . Notice, also, that $\frac{-a}{1+a} \in S$ because $\frac{-a}{1+a}$ is a real number (since $a \neq -1$) and $\frac{-a}{1+a}$ is not equal to -1 ; because if $\frac{-a}{1+a}$ were equal to -1 , then $\frac{-a}{1+a} = -1$, so $-a = -1-a$; that is, $0 = -1$.