Summer 1993 Exam 3

32

(1) Let x be an element of G with $x \neq e$. The hypothesis tells us that $\langle x \rangle$ must be G. Thus G is a cyclic group. If $\langle x \rangle$ were an infinite cyclic group, then $\{e\} \subsetneq \langle x^2 \rangle \subsetneq \langle x \rangle$. This contradicts the hypothesis; so G is a finite cyclic group. Let n be the order of G. If $n = n_1 \cdot n_2$ with $n_1 \neq 1$ and $n_2 \neq 1$ then $\{e\} \subsetneq \langle x^{n_1} \rangle \subsetneq \langle x \rangle$. Once again, the hypothesis has been contradicted. Thus G has prime order.

(2) No. Let $G = S_3$ $H = \langle (12) \rangle$ $a = (13)$ and $b = (123)$

Observe that
$$aH = \{(13), (13)(12) = (123)\}$$
$$bH = \{(123), (123)(12) = (13)\}$$
Equal
$$a^2H = H$$
$$b^2H = (132)H = \{(132), (132)(12) = (23)\}$$
Not equal

(3) Take a and $b \in G$

$$\gamma \circ \varphi(ab) = \gamma(\varphi(ab)) = \gamma(\varphi(a)\varphi(b)) = \gamma(\varphi(a)) \cdot \gamma(\varphi(b))$$

the definition of compose

φ is a homom

γ is a homom.

$$= (\gamma \circ \varphi)(a) \cdot (\gamma \circ \varphi)(b)$$

(4) $\varphi(gh) = x\,g\,h\,x^{-1} = (xgx^{-1})(xhx^{-1}) = \varphi(g) \cdot \varphi(h)$

(5) Yes The element $x = (w_3, w_5)$ has order 15, where $w_3 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$ and $w_5 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$.
We know that the order of x must divide 15, so order $x = 1$ or 3 or 5 or 10 or 15 or 6 or 9 or 12

but $x^1 \neq (1,1)$, $x^3 = (1, w_5^3) \neq (1,1)$, $x^5 = (w_3^2, 1) \neq (1,1)$

$x^6 = (1, w_5) \neq (1,1)$ $x^9 = (1, w_5^4) \neq (1,1)$ $x^{10} = (w_3^1, 1) \neq (1,1)$ $x^{12} = (1, w_5^2) \neq (1,1)$ $\therefore$ order $x = 15$