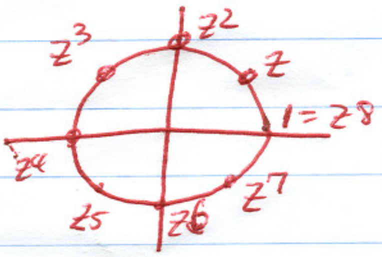
Summer 1993 Exam I

① The order of $z = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$ is 8 because
$z^8 = \cos 8(\frac{\pi}{4}) + i\sin 8(\frac{\pi}{4}) = \cos 2\pi + i\sin 2\pi = 1 + 0i = 1$
and no smaller power of z is equal to 1 as the picture shows



② $U_2 = \{1, -1\}$ $U_3 = \{1, \omega, \omega^2\}$ for $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

	$(1,1)$	$(1,\omega)$	$(1,\omega^2)$	$(-1,1)$	$(-1,\omega)$	$(-1,\omega^2)$
$(1,1)$	$(1,1)$	$(1,\omega)$	$(1,\omega^2)$	$(-1,1)$	$(-1,\omega)$	$(-1,\omega^2)$
$(1,\omega)$	$(1,\omega)$	$(1,\omega^2)$	$(1,1)$	$(-1,\omega)$	$(-1,\omega^2)$	$(-1,1)$
$(1,\omega^2)$	$(1,\omega^2)$	$(1,1)$	$(1,\omega)$	$(-1,\omega^2)$	$(-1,1)$	$(-1,\omega)$
$(-1,1)$	$(-1,1)$	$(-1,\omega)$	$(-1,\omega^2)$	$(1,1)$	$(1,\omega)$	$(1,\omega^2)$
$(-1,\omega)$	$(-1,\omega)$	$(-1,\omega^2)$	$(-1,1)$	$(1,\omega)$	$(1,\omega^2)$	$(+1,1)$
$(-1,\omega^2)$	$(-1,\omega^2)$	$(-1,1)$	$(-1,\omega)$	$(1,\omega^2)$	$(+1,1)$	$(1,\omega)$

The group $U_2 \times U_3$ is cyclic because
$(-1,\omega)^2 = (+1,\omega^2)$; $(-1,\omega)^3 = (-1,1)$; $(-1,\omega)^4 = (1,\omega)$
$(-1,\omega)^5 = (-1,\omega^2)$; $(-1,\omega)^6 = (1,1)$

3) $H = \{a \in \mid a^2 = e\}$

identity eH: $e^2 = e$ ✓ $\therefore e \in H$

closure: If $a, b \in H$, I must show $a * b \in H$.

$(a*b)^2 = abab = aabb = ee = e$ ✓
↑
G is abelian

inverses If $a \in H$, I must show $a^{-1} \in H$

but $a \cdot a = e \Rightarrow a = a^{-1}$ $\therefore a^{-1} \in H$

④ $K = \{ aha^{-1} \mid h \in H \}$

<u>$e \in K$</u> $e \in H$ thus $e = aea^{-1} \in K$

<u>closure</u> Take $k_1, k_2 \in K$ observe $k_1 = ah_1a^{-1}$
$k_2 = ah_2a^{-1}$

for some $h_i \in H$

then $k_1k_2 = ah_1a^{-1}ah_2a^{-1} = ah_1h_2a^{-1}$
and this is in K because $h_1h_2 \in H$

<u>inverses</u> If k is in K then $k = aha^{-1}$ for some $h \in H$, but $k^{-1} = ah^{-1}a^{-1}$ which is in K because $h^{-1} \in H$

⑤ Let $G = (GL_2(\mathbb{Z}), \times)$ $a = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$ $b = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}$

$a * b = \begin{pmatrix} 4 & 3 \\ 5 & 4 \end{pmatrix}$ $(a*b)^2 = \begin{pmatrix} 4 & 3 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ 5 & 4 \end{pmatrix} = \begin{pmatrix} 31 & 24 \\ 40 & 31 \end{pmatrix}$

$a^2 * b^2 = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ 9 & 7 \end{pmatrix}$

$= \begin{pmatrix} 30 & 23 \\ 43 & 33 \end{pmatrix}$