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9. Let G be the group of non-zero complex numbers under multiplication. Let G' be the group of non-zero 2×2 matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, with real entries, under multiplication. Consider the function $\varphi \colon G \to G'$, which is given by $\varphi(a+bi) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. Prove that φ is a group isomorphism.

φ respects the group operations

$$\varphi((a+bi)(c+di)) = \varphi((ac-bd)+i(bc+ad)) = \begin{bmatrix} ac-bd & bc+ad \\ -bc-ad & ac-bd \end{bmatrix}$$

$$\varphi(a+bi)\varphi(c+di) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \begin{bmatrix} ac-bd & ad+bc \\ -ad-bc & ac-bd \end{bmatrix}$$ these are the same.

φ is onto Take a typical element of G'. It is $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ with $a \neq 0$ or $b \neq 0$. This matrix $= \varphi(a+bi)$ and $a+bi \in G$.

φ is 1-1 suppose $a+bi$, $c+di \in G$, with $\varphi(a+bi) = \varphi(c+di)$

so $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} c & d \\ -d & c \end{bmatrix}$ so $a=c$ and $b=d$ so

$a+bi = c+di$.