10. Let $W = \mathbb{R} \setminus \{2\}$. Define $*$ on W by $a * b = ab - 2a - 2b + 6$. Prove that $(W, *)$ is a group.

31

Closure: If $a, b \in W$ then $a*b \in \mathbb{R}$. If $a*b$ happens to be 2 then $ab - 2a - 2b + 6 = 2$ so $ab - 2a - 2b + 4 = 0$ so $(a-2)(b-2) = 0$ so $a = 2$ or $b = 2$ and neither of these happened because $a, b \in \mathbb{R} \setminus \{2\}$.
So $a*b \neq 2$ and $a*b \in W$

assoc: Take $a, b, c \in W$

$(a*b)*c = (ab - 2a - 2b + 6)*c = (ab - 2a - 2b + 6)c - 2(ab - 2a - 2b + 6) - 2c + 6$

$= abc - 2ac - 2bc - 2ab + 4a + 4b + 4c - 6$

$a*(b*c) = a*(bc - 2b - 2c + 6) = a(bc - 2b - 2c + 6) - 2a - 2(bc - 2b - 2c + 6) + 6$

$= abc - 2ac - 2bc - 2ab + 4a + 4b + 4c - 6$

So $(a*b)*c = a*(b*c)$ and $*$ associates

3 is the identity element because

$a * 3 = 3a - 2a - 6 + 6 = a$

and $3 * a$ = exactly the same

inverses: Take $a \in W$. $a \neq 2$ so $\frac{2a-3}{a-2} \in \mathbb{R}$ also $\frac{2a-3}{a-2} \neq 2$ because $2a - 3 \neq 2(a-2)$. So $\frac{2a-3}{a-2} \in W$.

I claim that $\frac{2a-3}{a-2}$ is a's inverse

$$\frac{2a-3}{a-2} * a = \frac{2a-3}{a-2}a - 2\frac{2a-3}{a-2} - 2a + 6$$

$$= \frac{2a-3}{a-2}(a-2) - 2a + 6$$

$$= \cancel{2a} - 3 - \cancel{2a} + 6 = 3$$

$a * \frac{2a-3}{a-2}$ = exactly the same. Thus $\frac{2a-3}{a-2}$ is a's inverse and it is in W.

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