Math 546, Final Exam, Spring 2004, Solutions

PRINT Your Name:______________________________

There are 17 problems on 6 pages. The exam is worth 100 points.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**. Otherwise, get your course grade from VIP.

I will post the solutions on my website on **Wednesday**.

1. **(5 points) Define "centralizer". Use complete sentences.**

The *centralizer* of the element a in the group G is the set of all elements in G which commute with a.

2. **(5 points) Define "normal subgroup". Use complete sentences.**

The subgroup N of the group G is a *normal subgroup* if $gng^{-1} \in N$ for all $n \in N$ and all $g \in G$.

3. **(6 points) (Yes or No. If yes, PROVE it. If no, give a COUNTEREXAMPLE.) Let a and b be elements of finite order in the group G. Does ab have to have finite order?**

NO. Let G be the group of rigid motions of the xy plane, σ be reflection across the x-axis, and ρ be rotation by $\theta = \frac{2\pi}{\sqrt{2}}$ radians. Let $a = \sigma$ and $b = \sigma\rho$. It is clear that a has order 2. It is not hard to see that b is reflection across the line through the origin which makes the angle $\frac{-\theta}{2}$ with the positive x-axis; thus, b also has order 2. On the other hand, $ab = \rho$, which has infinite order; because, if ρ^m were equal to the identity for some positive integer m, then $m\theta = \frac{2m\pi}{\sqrt{2}}$ would equal an integer multiple of 2π and $\sqrt{2}$ would be a rational number.

4. **(6 points) Recall that each element of $\mathbb{C}$ is a point on the complex plane. Notice that $(\mathbb{R}^{\text{pos}}, \times)$ is a subgroup of $(\mathbb{C} \setminus \{0\}, \times)$. Give a geometric description of the left cosets of $(\mathbb{R}^{\text{pos}}, \times)$ in $(\mathbb{C} \setminus \{0\}, \times)$.**

The left cosets of $(\mathbb{R}^{\text{pos}}, \times)$ in $(\mathbb{C} \setminus \{0\}, \times)$ are the open rays emanating from the origin. Indeed, the left coset determined by $e^{i\theta}$ is the ray which forms the angle θ with the positive x-axis.

5. **(6 points) (Yes or No. If yes, PROVE it. If no, give a COUNTEREXAMPLE.) Let a be a fixed element of the group G. Consider the function $\rho_a \colon G \to G$, which is given by $\rho_a(g) = ga$, for all g in G. Is ρ_a onto?**

YES. Take an arbitrary element g in G. We see that $ga^{-1} \in G$ with $\rho_a(ga^{-1}) = g$.

6. **(6 points) (Yes or No. If yes, PROVE it. If no, give a COUNTEREXAMPLE.) Let a be a fixed element of the group G. Consider the function $\rho_a\colon G \to G$, which is given by $\rho_a(g) = ga$, for all g in G. Is ρ_a a homomorphism?**

NO! Let G be $(\mathbb{R}^{\text{pos}}, \times)$ and $a = 2$. We see that $\rho_2(1 \cdot 1) = \rho_2(1) = 2$. On the other hand, $\rho_2(1) \cdot \rho_2(1) = 2 \cdot 2 = 4 \neq 2$.

7. **(6 points) (Yes or No. If yes, PROVE it. If no, give a COUNTEREXAMPLE.) Is $\varphi\colon \mathbb{Z}_{10} \to \mathbb{Z}_5$, which is given by $\varphi([n]_{10}) = [n]_5$, a function?**

YES! If $[n]_{10} = [m]_{10}$, then 10 divides into $n - m$ evenly, so 5 also divides into $n - m$ evenly and $[n]_5 = [m]_5$.

8. **(6 points) (Yes or No. If yes, PROVE it. If no, give a COUNTEREXAMPLE.) Is $\varphi\colon \mathbb{Z}_5 \to \mathbb{Z}_{10}$, which is given by $\varphi([n]_5) = [n]_{10}$, a function?**

NO! Observe that $[0]_5 = [5]_5$, but $[0]_{10} \neq [5]_{10}$.

9. **(6 points) Let N be a normal subgroup of the group G, and let $\frac{G}{N}$ be the set of left cosets of N in G. Prove that $\varphi\colon \frac{G}{N} \times \frac{G}{N} \to \frac{G}{N}$, which is given by**

$$\varphi(aN, bN) = abN,$$

is a function.

If $aN = a'N$ and $bN = b'N$, then $a = a'n_1$ and $b = b'n_2$ for some n_1 and n_2 in N. We see that

$$ab = a'n_1b'n_2 = a'b'[(b')^{-1}n_1b']n_2 \in a'b'N,$$

since $(b')^{-1}n_1b'$ is an element of the normal subgroup N. It follows that $abN = a'b'N$.

10. **(6 points) (Yes or No. If yes, PROVE it. If no, give a COUNTEREXAMPLE.) Let $f\colon \mathbb{Z} \to \mathbb{Z}$ be a one-to-one and onto function. Suppose $B \subseteq \mathbb{Z}$ with $f(B) \subseteq B$. Is $f(B) = B$?**

Let B be the set of positive integers. Notice that the function $f\colon \mathbb{Z} \to \mathbb{Z}$, which is given by $f(n) = n + 1$, is a one-to-one and onto function which carries each element of B to another element of B. However, $f(B)$ is a proper subset of B, because $1 \in B$ and $f(b) \neq 1$ for any $b \in B$.

11. **(6 points) What is the order of $([2]_6, [2]_4) + <([3]_6, [2]_4)>$ in $\frac{\mathbb{Z}_6 \times \mathbb{Z}_4}{<([3]_6, [2]_4)>}$? Explain.**

Let x be the element $([2]_6, [2]_4) + <([3]_6, [2]_4)>$ of the group $G = \frac{\mathbb{Z}_6 \times \mathbb{Z}_4}{<([3]_6, [2]_4)>}$. We show that x has order 6 in G. We make our calculation in $\mathbb{Z}_6 \times \mathbb{Z}_4$. Let N be the subgroup

$$<([3]_6, [2]_4)> = \{([3]_6, [2]_4), ([0]_6, [0]_4)\}$$

of $\mathbb{Z}_6 \times \mathbb{Z}_4$ and let a be the element $([2]_6, [2]_4)$ of $\mathbb{Z}_6 \times \mathbb{Z}_4$. We show that the least positive integer n, with $\underbrace{a + \cdots + a}_{n}$ in N is 6. Notice that none of the elements

$$a = ([2]_6, [2]_4), \quad a + a = ([4]_6, [0]_4),$$

$$a+a+a = ([0]_6, [2]_4), \quad a+a+a+a = ([2]_6, [0]_4), \quad a+a+a+a+a = ([4]_6, [2]_4)$$

is in N; but

$$a+a+a+a+a+a = ([0]_6, [0]_4)$$

and this is in N.

12. **(6 points) Let H be a non-zero subgroup of $\mathbb{Z}$. Prove that H is cyclic.**

The subgroup H contains some element in addition to zero. Either this element or its inverse is positive. Let h_0 be the least positive element of H. We will show that $H = h_0\mathbb{Z}$. It is clear that $h_0\mathbb{Z} \subset H$. We complete the proof by showing that $H \subset h_0\mathbb{Z}$. Let h be an arbitrary element of H. Divide h_0 into h in order to obtain integers n and r with $h = nh_0 + r$ with $0 \le r < h_0$. We see that $r = h - nh_0$ is in H. The choice of h_0 (as the least positive element of H) forces r to be zero. Thus, $h \in h_0\mathbb{Z}$ and the proof is complete.

13. **(6 points) Let d be the greatest common divisor of the integers n and m. Prove that there exist integers r and s with $rn + sm = d$.**

Let $H = \{rn+sm \mid r, s \in \mathbb{Z}\}$. It is clear that H is a subgroup of $\mathbb{Z}$; hence, by the previous problem, H is cyclic and generated by some positive integer h_0. We will show that $h_0 = d$. Well, n and m are in H; so, h_0 is a common divisor of n and m. But, d is the greatest common divisor of n and m; hence, $h_0 \le d$. On the other hand, $h_0 \in H$; so, $h_0 = rn + sm$ for some integers r and s. We know that d divides n and m; so, d divides h_0. It follows that $d \le h_0$. Therefore, d must equal h_0.

14. **(6 points) List 6 subgroups of the Dihedral group D_4. No explanation is needed.**

Some of the subgroups of D_4 are:

$$D_4, \quad \{\text{id}\}, \quad \{id, \sigma, \sigma\rho^2, \rho^2\}, \quad \{\rho^2, \text{id}\}, \quad \{\sigma, \text{id}\}, \quad \{\sigma\rho, \text{id}\}, \quad \{\sigma\rho^2, \text{id}\}.$$

15. **(6 points) Prove that $(\mathbb{R}, +)$ is isomorphic to $(\mathbb{R}^{\text{pos}}, \times)$.**

Define $\varphi\colon (\mathbb{R}, +) \to (\mathbb{R}^{\text{pos}}, \times)$ by $\varphi(r) = e^r$. We see that φ is a homomorphism because, if $r, s \in \mathbb{R}$, then

$$\varphi(r + s) = e^{r+s} = e^r e^s = \varphi(r)\varphi(s).$$

We see that φ is onto. Let t be a positive real number. It follows that $\ln t$ is a real number with $\varphi(\ln t) = e^{\ln t} = t$. We see that φ is one-to-one. If r and s are real numbers with $\varphi(r) = \varphi(s)$, then $e^r = e^s$. Apply $\ln$ to both sides to see that $r = s$.

16. **(6 points) Consider $(\mathbb{Z}, *)$, where $n * m = n + m + 1$ for all integers n and m. Is $(\mathbb{Z}, *)$ a group? Explain.**

YES.
Closure: If n and m are in $\mathbb{Z}$, then $n * m = n + m + 1$ is also in $\mathbb{Z}$.
Identity: We see that -1 is the identity element because $(-1) * a = -1 + a + 1 = a$ for all a in $\mathbb{Z}$.
Inverses: If a is in $\mathbb{Z}$, then the inverse of a is $-a - 2$ because $a * (-a - 2) = a + (-a - 2) + 1 = -1$, which is the identity element.
Associativity: If a, b, and c are in $\mathbb{Z}$, then

$$a * (b * c) = a * (b + c + 1) = a + (b + c + 1) + 1 = a + b + c + 2$$

and

$$(a * b) * c = (a * b) * c = (a + b + 1) * c = (a + b + 1) + c + 1 = a + b + c + 2.$$

These values are equal; therefore, associativity holds.

17. **(6 points) S be a set and let B be a subset of S. Define**

$$H = \{\sigma \in \mathrm{Sym}(S) \mid \sigma(b) \in B \textbf{ for all } b \in B\}.$$

Suppose $S = \{1, 2, 3, 4, 5, 6\}$ and $B = \{1, 3, 5\}$. How many elements does H have? Explain.

If σ is in H, then $\sigma = \sigma' \sigma''$, where σ' is a permutation of $\{2, 4, 6\}$ and σ'' is a permutation of $\{1, 3, 5\}$. There are 6 choices for σ' and there are 6 choices for σ''. Thus, the group H has 36 elements.