Math 546, Spring 2004, Exam 4, Solutions

PRINT Your Name:_________________________________

There are 10 problems on 5 pages. The exam is worth 50 points. Each problem is worth 5 points.

I won't grade your exam until Monday. So don't be surprised if I don't e-mail your grade to you until then.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

If you would like, I will leave your exam outside my office after I have graded it. (If you like, I will send you an e-mail when I am finished with it.) You may pick it up any time between then and the next class. **Let me know if you are interested.**

I will post the solutions on my website on **Monday**.

1. **Write** $(1,4)(1,2,3,4,5)(4,6,7)$ **as a product of disjoint cycles.**

$\boxed{(1,2,3)(4,6,7,5)}$.

2. **Prove that the group of real numbers under addition is isomorphic to the group of positive real numbers under multiplication.**

Define $\varphi\colon(\mathbb{R},+)\to(\mathbb{R}^{\text{pos}},\times)$ by $\varphi(r)=e^r$. We see that φ is a homomorphism because, if $r,s\in\mathbb{R}$, then

$$\varphi(r+s)=e^{r+s}=e^re^s=\varphi(r)\varphi(s).$$

We see that φ is onto. Let t be a positive real number. It follows that $\ln t$ is a real number with $\varphi(\ln t)=e^{\ln t}=t$. We see that φ is one-to-one. If r and s are real numbers with $\varphi(r)=\varphi(s)$, then $e^r=e^s$. Apply $\ln$ to both sides to see that $r=s$.

Problems 3, 4, and 5 all refer to the following situation: Let S be a set and let B be a subset of S. Define

$$H=\{\sigma\in\operatorname{Sym}(S)\mid\sigma(b)\in B\text{ for all }b\in B\}.$$

3. **Suppose** $S=\{1,2,3,4,5\}$ **and** $B=\{2,3\}$**. LIST the elements of** H**.**

The elements of H are

$$(1),(14),(15),(45),(145),(154),(23),(23)(14),(23)(15),(23)(45),(23)(145),$$

and

$$(23)(154).$$

4. **Return to the general situation as as described before problem three. Assume that the set S is finite. Prove that H is a subgroup of $\mathrm{Sym}(S)$.**

The group $\mathrm{Sym}(S)$ is finite. It is clear that H is not empty since the identity permutaion is in H. It suffices to prove that H is closed. Take σ and τ from H. We know that $\sigma \circ \tau \in \mathrm{Sym}(S)$. We must show that $\sigma \circ \tau \in H$. Suppose $b \in B$. Observe that $(\sigma \circ \tau)(b) = \sigma(\tau(b))$. I know that $\tau(b)$ is in B because τ is in H. I know that σ carries every element of B back into B because σ is in H. It follows that σ of the element $\tau(b)$ of B, is in B. Thus, $(\sigma \circ \tau)(b) \in B$ for all b in B and $\sigma \circ \tau \in H$.

5. **Return to the general situation as described before problem three. Assume that the set S is infinite. Give an example in which H is NOT a subgroup of $\mathrm{Sym}(S)$. Explain your example thoroughly.**

Let S be the set of integers and B be the set of positive integers. Notice that the function $\sigma(n) = n + 1$ is a permutation of S which carries each element of B to another element of B. The inverse of σ in $\mathrm{Sym}(S)$ is σ^{-1}, where $\sigma^{-1}(n) = n - 1$. But σ^{-1} is NOT in H because $1 \in B$ and $\sigma^{-1}(1) = 0 \notin B$.

6. **Let G be a group and a be a fixed element of G. Define $\phi\colon G \to G$ by $\phi(g) = aga^{-1}$ for all $g \in G$. Prove that ϕ is a group isomorphism.**

We see that ϕ is a homomorphism. If g and h are in G, then

$$\phi(gh) = agha^{-1} = aga^{-1}aha^{-1} = \phi(g)\phi(h).$$

We see that ϕ is onto. Take $g \in G$. Observe that $a^{-1}ga$ is an element of G with $\phi(a^{-1}ga) = aa^{-1}gaa^{-1} = g$. We see that ϕ is one-to-one. If g and h are in G with $\phi(g) = \phi(h)$, then $aga^{-1} = aha^{-1}$. Multiply both sides of the equation on the left by a^{-1} and on the right by a to see that $g = h$.

7. **Give two non-isomorphic groups of order 36. Explain why the groups are not isomorphic.**

The groups $\mathbb{Z}_{36}$ and $\mathbb{Z}_6 \times \mathbb{Z}_6$ each have 36 elements. These groups are not isomorphic because the element $[1]_{36}$ of $\mathbb{Z}_{36}$ has order 36 and every element of the group $\mathbb{Z}_6 \times \mathbb{Z}_6$ has order 6 or less.

8. **List the elements of the group $S_3 \times \mathbb{Z}_2$. What is the order of each element?**

element	order
$((1), [0]_2)$	1
$((12), [0]_2)$	2
$((13), [0]_2)$	2
$((23), [0]_2)$	2
$((123), [0]_2)$	3
$((132), [0]_2)$	3
$((1), [1]_2)$	2
$((12), [1]_2)$	2
$((13), [1]_2)$	2
$((23), [1]_2)$	2
$((123), [1]_2)$	6
$((132), [1]_2)$	6

9. **Exhibit an isomorphism $\phi\colon U \to G$, where U is the unit circle group and G is a subgroup of $\mathrm{GL}_2(\mathbb{R})$. Tell me what G is. Tell me what ϕ is. Prove that ϕ is an isomorphism.**

Let G be the subgroup
$$\left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \middle| \, a^2+b^2=1 \right\}$$
of $\mathrm{GL}_2(\mathbb{R})$. Define $\phi\colon U \to G$ by $\phi(a+b\imath) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. Notice that if $z = a+b\imath \in U$, then $a^2+b^2=1$; so $\phi(z)$ really is in G.

We see that ϕ is a homomorphism. Take $z=a+b\imath$ and $w=c+d\imath$ from U. Observe that
$$\phi(zw) = \phi(ac-bd+\imath(ad+bc)) = \begin{bmatrix} ac-bd & -ad-bc \\ ad+bc & ac-bd \end{bmatrix}.$$
On the other hand,
$$\phi(z)\phi(w) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac-bd & -ad-bc \\ ad+bc & ac-bd \end{bmatrix}.$$
Thus, $\phi(zw)=\phi(z)\phi(w)$ and ϕ is a homomorphism.

We see that ϕ is one-to-one. Take $z=a+b\imath$ and $w=c+d\imath$ from U with $\phi(z)=\phi(w)$. Thus, the matrices
$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c & -d \\ d & c \end{bmatrix}$$
are equal. So the corresponding entries are equal. That is, $a=c$ and $b=d$. Thus, $z=w$.

We see that ϕ is onto. Take $M = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ from G. Let $z=a+b\imath$. Notice that $z \in U$ (since $a^2+b^2=1$) and $\phi(z)=M$.

10. **Exhibit an isomorphism** $\phi\colon (\mathbb{R}\setminus\{0\},\times)\to(\mathbb{R}\setminus\{-2\},*)$ **, where** $a*b=ab+2a+2b+2$ **. Tell me what** ϕ **is and prove that** ϕ **is an isomorphism.**

Define $\phi\colon (\mathbb{R}\setminus\{0\},\times)\to(\mathbb{R}\setminus\{-2\},*)$ by $\phi(r)=r-2$.
We see that ϕ **is a homomorphism.** If r and s are in $\mathbb{R}\setminus\{0\}$, then

$$\phi(rs)=rs-2.$$

On the other hand,

$$\phi(r)*\phi(s)=(r-2)*(s-2)=(r-2)(s-2)+2(r-2)+2(s-2)+2$$

$$=rs-2s-2r+4+2r-4+2s-4+2=rs-2.$$

Thus, $\phi(rs)=\phi(r)*\phi(s)$ and ϕ is a homomorphism.
We see that ϕ **is a onto.** Take $t\in\mathbb{R}\setminus\{-2\}$. Observe that $t+2\in\mathbb{R}\setminus\{0\}$ and $\phi(t+2)=t$.
We see that ϕ **is one-to-one.** If r and s are elements of $\mathbb{R}\setminus\{0\}$ with $\phi(r)=\phi(s)$, then $r-2=s-2$. Add 2 to both sides to see that $r=s$.