Math 546, Exam 1, Spring, 2004 Solutions

PRINT Your Name:________________________________

There are 8 problems on 5 pages. Problems 1 and 2 are worth 7 points each. Each of the other problems is worth 6 points.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

If you would like, I will leave your exam outside my office tomorrow by about noon, you may pick it up any time between then and the next class. **Let me know if you are interested.**

I will post the solutions on my website at about 6:30 PM today.

1. **Define "group". Use complete sentences.**

A *group* is a set G together with an operation $*$ which satisfies the following properties.
Closure: If a and b are elements of G, then $a * b$ is an element of G.
Associativity: If a, b, and c are elements of G, then $(a * b) * c = a * (b * c)$.
Identity element: There exists an element id in G with $\text{id} * a = a$ and $a * \text{id} = a$ for all a in G.
Inverses: If a is in G, then there is an element b in G with $a * b = \text{id}$ and $b * a = \text{id}$.

2. **Exhibit a group G and two elements a and b of G with $(ab)^2 \neq a^2b^2$.**

Let G be the group D_3. (See problem 8, if necessary.) Let $a = \sigma$ (which is relflection across the x-axis) and $b = \sigma\rho$ (which is reflection across the line $y = -\sqrt{3}x$). We see that

$$(ab)^2 = (\sigma\sigma\rho)^2 = \rho^2$$

and this is rotation ccw by 240 degrees. On the other hand,

$$a^2b^2 = \sigma^2(\sigma\rho)^2 = \text{id}\,\text{id} = \text{id},$$

since every reflection squares to become the identity function. We conclude that $(ab)^2 \neq a^2b^2$.

3. **Let G be a group and let H and K be subgroups of G. Is the intersection $H \cap K$ always a subgroup of G? If yes, prove the result. If no, show a counterexample.**

Yes.
Closure. If a and b are in $H \cap K$, then a and b are in H; hence, $a * b$ is in H, since H is a group. Also, a and b are in K; hence, $a * b$ is in K, since K is a group. Therefore, $a * b$ is in $H \cap K$.
Associativity holds for all products in G; hence, associativity holds on the smaller set $H \cap K$.
Identity. The identity element id of G is an element of each of the subgroups H and K; hence, id is in $H \cap K$.

Inverses. If a is an element of $H \cap K$, then the inverse of a in G is an element of H because H is a subgroup of G . In a similar manner, the inverse of a in G is in K because K is a subgroup of G . Therefore, the inverse of a in G is in $H \cap K$.

4. **Let G be a group and let H and K be subgroups of G . Is the union $H \cup K$ always a subgroup of G ? If yes, prove the result. If no, show a counterexample.**

NO.
Consider $G = D_3$ as in problems 2 or 8. The subset $H = \{\text{id}, \sigma\}$ is a subgroup of G and the subset $K = \{\text{id}, \sigma\rho\}$ is a subgroup of G since both sets are closed under composition and the formation of inverses since every reflection squares to the identity function. However, the union

$$H \cup K = \{\text{id}, \sigma, \sigma\rho\}$$

is not a subgroup of G since this set is not closed: $\sigma \in H \cup K$, $\sigma\rho \in H \cup K$, but the composition

$$\sigma(\sigma\rho) = \rho$$

is not in $H \cup K$.

5. **Let $S = \mathbb{R} \setminus \{-2\}$. Define $*$ on S by $a * b = ab + 2a + 2b + 2$. Prove that $(S, *)$ is a group.**

Closure: Take a, b from S . We must show that $a * b$ is in S . Well, $a * b = ab + 2a + 2b + 2$, which is clearly a real number. We must check that $ab + 2a + 2b + 2$ is not equal to -2 . If $ab + 2a + 2b + 2$ were equal to -2 , then $ab + 2a + 2b + 2 = -2$; so, $ab + 2a + 2b + 4 = 0$; that is, $(a + 2)(b + 2) = 0$; so $a = -2$ or $b = -2$. On the other hand, a and b are in S ; so neither a nor b is -2 . We conclude that $ab + 2a + 2b + 2 \neq -2$; therefore, $ab + 2a + 2b + 2 \in S$.

Associativity: Take a , b , and c from S . Observe that

$$a * (b * c) = a * (bc + 2b + 2c + 2) = a(bc + 2b + 2c + 2) + 2a + 2(bc + 2b + 2c + 2) + 2$$

$$= abc + 2(ab + ac + bc) + 4(a + b + c) + 6.$$

On the other hand,

$$(a * b) * c = (ab + 2a + 2b + 2) * c = (ab + 2a + 2b + 2)c + 2(ab + 2a + 2b + 2) + 2c + 2$$

$$= abc + 2(ab + ac + bc) + 4(a + b + c) + 6.$$

We see that $a * (b * c) = (a * b) * c$.

Identity: The number -1 is the identity element of S because $a * (-1) = a(-1) + 2a + 2(-1) + 2 = a$ and $(-1) * a = (-1)a + 2(-1) + 2a + 2 = a$ for all $a \in S$.

Inverses: Take $a \in S$. The inverse of a is $\frac{-3-2a}{a+2}$ because

$$a * \frac{-3-2a}{a+2} = a\frac{-3-2a}{a+2} + 2a + 2\frac{-3-2a}{a+2} + 2 = \frac{(a+2)(-3-2a)}{a+2} + 2a + 2 =$$

$$-3 - 2a + 2a + 2 = -1.$$

The operation * is commutative; so, $\frac{-3-2a}{a+2} * a$ is also equal to 0. Notice, also, that $\frac{-3-2a}{a+2} \in S$ because $\frac{-3-2a}{a+2}$ is a real number (since $a \neq -2$) and $\frac{-3-2a}{a+2}$ is not equal to -2; because if $\frac{-3-2a}{a+2}$ were equal to -2, then $\frac{-3-2a}{a+2} = -2$, so $-3 - 2a = -2a - 4$; that is, $-3 = -4$.

6. **Define "centralizer". Use complete sentences.**

Let g be an element in the group G. The *centralizer* of g in G is the set of all elements of G which commute with g.

7. **Let $G = \{\left[\begin{smallmatrix} a & c \\ 0 & b \end{smallmatrix}\right] \mid a, b, c \in \mathbb{R}$ with $a \neq 0$ and $b \neq 0\}$. The set G forms a group under matrix multiplication. (You do not have to prove this.) Find the centralizer of $g = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ in G.**

I look for $\{M \in G \mid gM = Mg\}$. A typical element M in G has the form $M = \left[\begin{smallmatrix} a & c \\ 0 & b \end{smallmatrix}\right]$, with $a, b, c \in \mathbb{R}$, $a \neq 0$, and $b \neq 0$. For such a matrix M,

$$gM = \begin{bmatrix} a & b+c \\ 0 & b \end{bmatrix} \quad \text{and} \quad Mg = \begin{bmatrix} a & a+c \\ 0 & b \end{bmatrix}.$$

We conclude that M is in the centralizer of g if and only if $a = b$. In other words, the centralizer of g is

$$\{\left[\begin{smallmatrix} a & c \\ 0 & a \end{smallmatrix}\right] \mid a, c \in \mathbb{R} \text{ with } a \neq 0 \}.$$

8. **Recall that D_3 is the smallest subgroup of the group of rigid motions which contains ρ and σ, where ρ is rotation counter clockwise by $120°$ fixing the origin and σ is reflection of the xy plane across the x axis. Recall also that the elements of D_3 are: id, ρ, ρ^2, σ, $\sigma\rho$, and $\sigma\rho^2$. Let H be the following subset of D_3:**

$$H = \{g^3 \mid g \in D_3\}.$$

   (a) List the elements of H.
   (b) Is H a subgroup of D_3? Explain.

The elements of H are

$$\text{id}^3 = \text{id}, \ \rho^3 = \text{id}, \ (\rho^2)^3 = \text{id}, \ \sigma^3 = \sigma, \ (\sigma\rho)^3 = \sigma\rho, \quad \text{and} \quad (\sigma\rho^2)^3 = \sigma\rho^2.$$

That is,

$$H = \{\text{id}, \sigma, \sigma\rho, \sigma\rho^2\}.$$

The set H is NOT a subgroup of D_3 because H is not closed since $\sigma \in H$ and $\sigma\rho \in H$, but the product of $\sigma(\sigma\rho) = \rho$ is not in H.