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3. Let G be a group and $a \in G$.

(a) Define "the centralizer of a".

The centralizer of the element a in the group G is the set of all elements in G which commute with a.

(b) Prove that the centralizer of a is a subgroup of G.

Closure Take x and y in the centralizer of a. So $xa = ax$ and $ya = ay$. We now show that xy is in the centralizer of a. Well

$$(xy)a \underset{\text{assoc.}}{=} x(ya) \underset{y \in C(a)}{=} x(ay) \underset{\text{assoc}}{=} (xa)y \underset{x \in C(a)}{=} (ax)y = a(xy).$$

Thus xy is in the centralizer of a.

Inverses Assume x is in the centralizer of a. Then $xa = ax$. Multiply by x^{-1}: $x^{-1}xa = x^{-1}ax$, so $a = x^{-1}ax$. Multiply by x^{-1} so $ax^{-1} = x^{-1}a$. Thus x^{-1} is in the centralizer of a.

The identity element of G commutes with a. So the identity element of G is in the centralizer of a.

(c) Let $G = D_4$ and $a = \rho$. Find the centralizer of a.

It is clear that id, ρ, ρ^2 and ρ^3 all commute with ρ. So all 4 of these elements are in the centralizer of ρ. The centralizer of ρ is a subgroup of D_4 so Lagrange says that its order divides 8. So the centralizer of ρ has either 4 or 8 elements.

On the other hand $\rho\sigma = \sigma\rho^3 \neq \sigma\rho$ so $\sigma \notin C(\rho)$

Thus $C(\rho) = \langle \rho \rangle$

I used $C(a)$ to equal the centralizer of a.