Please PRINT your name ______________________________________

No calculators, cell phones, computers, notes, etc.

Make your work correct, complete and coherent.

Please take a picture of your quiz (for your records) just before you turn the quiz in. I will e-mail your grade and my comments to you. I will keep your quiz.

The quiz is worth 5 points. The solutions will be posted on my website later today.

Quiz 5, October 20, 2022

Let a and b be elements of a group G. Suppose that a and b both have finite order that the orders of a and b are relatively prime. Suppose further that $ab = ba$. Prove that the order of ab is equal to the order of a times the order of b. Recall that the *order* of a group element a is the least positive integer n with a^n equal to the identity element.

Answer: Let n be the order of a and m be the order of b. Use the hypothesis that a and b commute to see that

$$(ab)^{nm} = \underbrace{(ab)(ab)\cdots(ab)}_{nm \text{ times}} = a^{mn}b^{mn} = (a^n)^m(b^m)^n = \text{id}\,.$$

Now we must show that nm is the smallest positive power which sends ab to the identity. Suppose the order of ab is c. The elements a and b commute; hence

$$a^c b^c = (ab)^c = \text{id};$$

thus,

$$a^c = b^{-c} \in \langle a \rangle \cap \langle b \rangle.$$

The order of a^c divides the order of $\langle a \rangle$, which is n. The order of a^c also divides the order of $\langle b \rangle$, which is m. Hence, the order of a^c is a common divisor of n and m. On the other hand, the greatest common divisor of m and n is 1. Thus a^c has order 1; in other words, a^c is equal to the identity and n divides c.

In a similar manner, b^{-c} (which equals a^c) is also the identity; hence b^c is equal to the identity; so m divides c.

The numbers n and m are relatively prime and both numbers divide c. It follows that c is at least as large as nm.