Math 546, Exam 3, Fall, 2004

The exam is worth 50 points.

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Take enough space for each problem. Turn in your solutions in the order: problem 1, problem 2, ... ; although, by using enough paper, you can do the problems in any order that suits you.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will leave your exam outside my office TOMORROW by about 6PM, you may pick it up any time between then and the next class.

I will post the solutions on my website at about 4:00 PM today.

1. **(6 points) Define "normal subgroup". Use complete sentences.**

The subgroup N of the group G is a *normal subgroup* if $gng^{-1} \in N$ for all $g \in G$ and $n \in N$.

2. **(6 points) Define "cyclic group". Use complete sentences.**

The group G is a *cyclic group* if there exists an element $g \in G$ so that every element of G has the form g^n for some integer n.

3. **(6 points) Define "generator". Use complete sentences.**

The element g of the group G is a *generator* of G if every element of G has the form g^n for some integer n.

4. **(6 points) What is the order of $(1,1)+H$ in the group $\bar{G} = \frac{G}{H}$, where $G = \mathbb{Z}_6 \times \mathbb{Z}_4$ and $H = \{(0,0),(3,0),(0,2),(3,2)\}$? Is $\bar{G}$ a cyclic group? (A small amount of explanation is needed.)**

Let x be the element $(1,1)$ of G. Observe that the least positive integer n with $nx \in H$ is $n = 6$. Thus, the order of $x + H$ in $\bar{G}$ is 6. The group $\bar{G}$ has 6 elements, so $\bar{G}$ IS cyclic. Here is our calculation:

$$x = (1,1) \notin H, \quad 2x = (2,2) \notin H, \quad 3x = (3,3) \notin H, \quad 4x = (4,4) \notin H,$$
$$5x = (5,5) \notin H, \quad 6x = (6,6) = (0,2) \in H.$$

5. **(6 points) Find 2 distinct elements of order 2 in the group $\bar{G} = \frac{\mathbb{Z}_6 \times \mathbb{Z}_4}{<(2,2)>}$. Is $\bar{G}$ a cyclic group? (A small amount of explanation is needed.)**

Let G be the group $\mathbb{Z}_6 \times \mathbb{Z}_4$, H be the subgroup $<(2,2)>$ of G, x be the element $(1,0)$ of G, and y be the element $(0,1)$ of G. We see that

$$H = \{(0,0),(2,2),(4,0),(0,2),(2,0),(4,2)\}.$$

We also see that

$$x \notin H, \quad y \notin H, \quad x - y \notin H, \quad 2x \in H, \quad 2y \in H.$$

It follows that $x + H$ and $y + H$ are distinct elements of $\bar{G}$ of order 2.

6. **(7 points)**
   (a) Find an element of order 3 in $\frac{\mathbb{R}}{\mathbb{Z}}$. (A small amount of explanation is needed.)
   (b) Find an element of infinite order in $\frac{\mathbb{R}}{\mathbb{Z}}$. (A small amount of explanation is needed.)

The coset $1/3+\mathbb{Z}$ of $\frac{\mathbb{R}}{\mathbb{Z}}$ has order 3, because $1/3 \notin \mathbb{Z}$, $2/3 \notin \mathbb{Z}$, but $3/3 \in \mathbb{Z}$. The coset $\pi+\mathbb{Z}$ of $\frac{\mathbb{R}}{\mathbb{Z}}$ has infinite order because $n\pi \notin \mathbb{Z}$ for any positive integer n.

7. **(7 points)**
   (a) Does there exist a function $\varphi\colon \frac{\mathbb{Z}}{9\mathbb{Z}} \to \frac{\mathbb{Z}}{3\mathbb{Z}}$ with $\varphi(a+9\mathbb{Z}) = a+3\mathbb{Z}$ for all integers a? Explain thoroughly.
   (b) Does there exist a function $\varphi\colon \frac{\mathbb{Z}}{3\mathbb{Z}} \to \frac{\mathbb{Z}}{9\mathbb{Z}}$ with $\varphi(a+3\mathbb{Z}) = a+9\mathbb{Z}$ for all integers a? Explain thoroughly.

   (a) Yes, φ IS a function. If the cosets $a+9\mathbb{Z}$ and $b+9\mathbb{Z}$ are equal, then $a-b \in 9\mathbb{Z} \subseteq 3\mathbb{Z}$; hence, the cosets $a+3\mathbb{Z}$ and $b+3\mathbb{Z}$ are equal.
   (b) NO, φ is NOT a function! We see that the cosets $0+3\mathbb{Z}$ and $3+3\mathbb{Z}$ are equal. We also see that the cosets $0+9\mathbb{Z}$ and $3+9\mathbb{Z}$ are NOT equal. There is no FUNCTION which sends $0+3\mathbb{Z}$ to $0+9\mathbb{Z}$ and $3+3\mathbb{Z}$ to $3+9\mathbb{Z}$.

8. **(6 points) Let K be a subgroup of the group G and let N be a normal subgroup of G. Prove that**

$$H = \{kn \mid k \in K \text{ and } n \in N\}$$

   is a subgroup of G.

Closure: Take h_1 and h_2 from H. We know that $h_1 = k_1n_1$ and $h_2 = k_2n_2$ for some $k_i \in K$ and $n_i \in N$. Observe that

$$h_1h_2 = k_1n_1k_2n_2 = k_1k_2(k_2^{-1}n_1k_2)n_2.$$

We know that $k_1k_2 \in K$ because K is a group and $(k_2^{-1}n_1k_2)n_2 \in N$ because N is a normal subgroup of G. Thus, $h_1h_2 \in H$ and H is closed.

Identity: Let id_G be the identity element of G, id_K the identity element of K, and id_N the identity element of N. We know $\mathrm{id}_G = \mathrm{id}_K\mathrm{id}_N$ because all three identity elements are equal. Thus, the identity element of G is in H.

Inverses: Take $h = kn$ from H, with $k \in K$ and $n \in N$. We know that $h^{-1} = n^{-1}k^{-1} = k^{-1}(kn^{-1}k^{-1})$. Furthermore, $k^{-1} \in K$ because K is a group and $(kn^{-1}k^{-1}) \in N$ because N is a normal subgroup of G. We conclude that h^{-1} is in H.