Math 546, Exam 2, Fall, 2004

The exam is worth 50 points.

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Take enough space for each problem. Turn in your solutions in the order: problem 1, problem 2, ... ; although, by using enough paper, you can do the problems in any order that suits you.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will leave your exam outside my office TOMORROW by about 5PM, you may pick it up any time between then and the next class.

I will post the solutions on my website at about 4:00 PM today.

1. **(6 points) Define "subgroup". Use complete sentences.**

The subset H of the group $(G,*)$ is a *subgroup* of G, if H is a group under the same operation $*$.

2. **(6 points) Define the "center of a group". Use complete sentences.**

The *center* of the group G is the set of all elements in G which commute with every element in G.

3. **(6 points) STATE Lagrange's Theorem.**

If H is a subgroup of the finite group G, then the order of H divides the order of G.

4. **(7 points) Let G be a finite group with an even number of elements. Prove that there must exist an element $a \in G$ with $a \neq \mathbf{id}$, but $a^2 = \mathbf{id}$.**

Observe that G is the disjoint union of the sets

$$Y = \{g \in G \mid g^2 = \mathrm{id}\} \quad \text{and} \quad N = \{g \in G \mid g^2 \neq \mathrm{id}\}.$$

The set Y always contains at least one element, namely id. Observe that if $g \in N$, then g^{-1} is also in N and $g \neq g^{-1}$. It follows that N may be partitioned into a collection of subsets each of which consists of a pair of elements which are inverses of one another. Thus, N contains an even number of elements. The hypothesis ensures that the group G contains an even number of elements. We conclude that Y contains an even number of elements. Since Y contains at least one element, we now know that Y must contain at least two elements. In other words, there does exist an element g in G with $g \neq \mathrm{id}$, but $g^2 = \mathrm{id}$.

5. **(7 points) Give an example of a finite group G and a proper subgroup H of G, with H not a cyclic group.**

Cosider $H = \{id, \sigma, \rho^2, \rho^2\sigma\}$ and $G = D_4$. We have seen that H is a subgroup of G. (If you like, H is equal to the centralizer of σ.) It is clear H is not cyclic because every element of H squares to the identity element.

6. **(6 points) Let G be a group of order pq where p and q are prime numbers. Prove that every proper subgroup of G is cyclic.**

If H is a proper subgroup of G and H is larger than $\{\text{id}\}$, then Lagrange's Theorem ensures that H has order p or q. Furthermore, Lagrange's Theorem also ensures that every group of prime order is cyclic.

7. **(6 points) Let g be an element of the group G. Suppose that G has order n. Prove that $g^n = \textbf{id}$.**

Let m equal the order of the cyclic subgroup $<g>$ of G. It follows that $g^m = \text{id}$. Lagrange's Theorem ensures that $m|n$; that is $n = mr$ for some integer r. We have $g^n = g^{mr} = (g^m)^r = (\text{id})^r = \text{id}$.

8. **(6 points) Let H be a subgroup of a group. Suppose that $g^{-1}hg \in H$ for all $g \in G$ and $h \in H$. Fix an element $g \in G$. Prove that $gH = Hg$, where gH is the LEFT coset**

$$gH = \{gh \mid h \in H\}$$

and Hg is the RIGHT coset

$$Hg = \{hg \mid h \in H\}.$$

$gH \subseteq Hg$: Take a typical element x of the coset gH. Thus, $x = gh$ for some $h \in H$. It follows that $x = gh = ghg^{-1}g$. The hypothesis ensures us that ghg^{-1} is an element of H; and therefore, $x = (ghg^{-1})g$ is an element of Hg.

$Hg \subseteq gH$: Take a typical element x of the coset Hg. Thus, $x = hg$ for some $h \in H$. It follows that $x = hg = gg^{-1}hg = g\left(g^{-1}h(g^{-1})^{-1}\right)$. The hypothesis ensures that $g^{-1}h(g^{-1})^{-1}$ is an element of H; therefore, $x \in gH$.