Math 546, Exam 1, Fall, 2004 Solutions

The exam is worth 50 points.

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Take enough space for each problem. Turn in your solutions in the order: problem 1, problem 2, ... ; although, by using enough paper, you can do the problems in any order that suits you.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will leave your exam outside my office tomorrow by about noon, you may pick it up any time between then and the next class.

I will post the solutions on my website at about 4:00 PM today.

1. **(6 points) Define "group". Use complete sentences.**

A *group* is a set G together with an operation $*$ which satisfies the following properties.
Closure: If a and b are elements of G, then $a * b$ is an element of G.
Associativity: If a, b, and c are elements of G, then $(a * b) * c = a * (b * c)$.
Identity element: There exists an element id in G with $\mathrm{id} * a = a$ and $a * \mathrm{id} = a$ for all a in G.
Inverses: If a is in G, then there is an element b in G with $a * b = \mathrm{id}$ and $b * a = \mathrm{id}$.

2. **(6 points) Define "centralizer". Use complete sentences.**

Let g be an element in the group G. The *centralizer* of g in G is the set of all elements of G which commute with g.

3. **(6 points) Exhibit a group G and two elements a and b of G with $(ab)^3 \neq a^3b^3$.**

Let $G = D_4$, $a = \rho$, and $b = \sigma$. We see that

$$(ab)^3 = (\rho\sigma)^3 = \rho\sigma$$

since $\rho\sigma$ is a reflection. On the other hand,

$$a^3b^3 = \rho^3\sigma^3 = \rho^3\sigma$$

and $\rho\sigma \neq \rho^3\sigma$.

4. **(7 points) Let G be a group and let a and b be elements of G with $a^{-1}b^{-1} = (ab)^{-1}$. Prove that $ab = ba$.**

Multiply both sides of $a^{-1}b^{-1} = (ab)^{-1}$ on the right by ab to see that

$$a^{-1}b^{-1}ab = (ab)^{-1}ab.$$

The fact that $(ab)^{-1}$ is the inverse of ab ensures that the right side of the above equation is id; so,

$$a^{-1}b^{-1}ab = \mathrm{id}.$$

Now multiply both sides of the above equation on the left by ba to get

$$baa^{-1}b^{-1}ab = ba\mathrm{id};$$

which yields $ab = ba$.

5. **(7 points) Let G be an abelian group and let H be the subset $H = \{g \in G \mid g^3 = \mathrm{id}\}$ of G. Prove that H is a subgroup of G.**

closure Take h_1 and h_2 from H. We see that $(h_1h_2)^3 = h_1^3h_2^3$, because G is abelian. On the other hand, each h_i is in H; so, each $h_i^3 = \mathrm{id}$; so, $(h_1h_2)^3 = h_1^3h_2^3 = (\mathrm{id})\mathrm{id} = \mathrm{id}$ and $h_1h_2 \in H$.
identity We see that $\mathrm{id}^3 = \mathrm{id}$; thus, $\mathrm{id} \in H$.
inverses Take $h \in H$. We know that h^{-1} exists in G. We must show that h^{-1} is in H. Multiply both sides of $h^3 = \mathrm{id}$ by $h^{-1}h^{-1}h^{-1}$ to see that $h^{-1}h^{-1}h^{-1}h^3 = h^{-1}h^{-1}h^{-1}$; or $\mathrm{id} = \left(h^{-1}\right)^3$. Thus, h^{-1} is in H.
associativity The operation associates on all of G; so the operation continues to associate on the subset H of G.

6. **(6 points) List the elements that are in the centralizer of $\rho^3\sigma$ in D_4. (I do not need to see any explanation.)**

$$\mathrm{id}, \rho^3\sigma, \rho\sigma, \rho^2$$

7. **(6 points) Let $G = (\mathbb{Q} \setminus \{0\}, *)$, where $a * b = |ab|$. Is G a group? Explain.**

No. This thing does not have an identity element. If id were an identity element, then

$$-1 = (-1) * \mathrm{id} = |-\mathrm{id}|.$$

I do not know what $-\mathrm{id}$ is; but I do know that $0 < |-\mathrm{id}|$; and therefore, $|-\mathrm{id}|$ does not equal -1.

8. **(6 points) Let $G = (\mathbb{Q}\setminus\{0\}, *)$, where $a*b = \frac{a}{b}$. Is G a group? Explain.**

No. The operation $*$ does not associate. Observe that

$$1 * (2 * 3) = \frac{1}{2 * 3} = \frac{1}{\frac{2}{3}} = \frac{3}{2};$$

but

$$(1 * 2) * 3 = \left(\frac{1}{2}\right) * 3 = \frac{\frac{1}{2}}{3} = \frac{1}{6}$$

and $\frac{3}{2} \neq \frac{1}{6}$.