Math 546, Exam 3, Spring 2010

Write everything on the blank paper provided.

You should KEEP this piece of paper.

If possible: turn the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. There are **7** problems.

Write **coherently** in **complete sentences**.

No Calculators or Cell phones.

I will post the solutions later today.

1. **(7 points) State Lagrange's Theorem. Use complete sentences. Write everything that is necessary, but nothing extra.**

If H is a subgroup of a finite group G, then the number of elements in H divides the number of elements in G.

2. **(7 points) Define *normal subgroup*. Use complete sentences. Write everything that is necessary for your definition to make sense, but nothing extra.**

The subgroup H of the group G is a *normal subgroup* if $ghg^{-1} \in H$ for all $g \in G$ and all $h \in H$.

3. **(7 points) Define *group homomorphism*. Use complete sentences. Write everything that is necessary for your definition to make sense, but nothing extra.**

The function φ from the group $(G, *)$ to the group $(G', *')$ is a group homomorphism if $\varphi(g_1 * g_2) = \varphi(g_1) *' \varphi(g_2)$ for all $g_1, g_2 \in G$.

4. **(8 points) Let H be a normal subgroup of the group G. Suppose that a, b, c, and d are elements of G with $aH = bH$ and $cH = dH$. Prove that $acH = bdH$.**

The hypothesis $aH = bH$ tells us that there exists an element $h_1 \in H$ with $a = bh_1$. The hypothesis $cH = dH$ tells us that there exists an element $h_2 \in H$ with $c = dh_2$.

We now show $acH \subseteq bdH$**.** A typical element of acH is ach for some $h \in H$. We see that

$$ach = (bh_1)(dh_2)h = bd(d^{-1}h_1d)h_2h.$$

The hypothesis that H is a normal subgroup of G tells us that $d^{-1}h_1d \in H$; therefore, $(d^{-1}h_1d)h_2h \in H$, since H is a group. We have shown that ach is equal to bd times an element of H; and therefore, $ach \in bdH$.

We now show $bdH \subseteq acH$**.** A typical element of bdH is bdh for some $h \in H$. We see that

$$bdh = (ah_1^{-1})(ch_2^{-1})h = ac(c^{-1}h_1^{-1}c)h_2^{-1}h.$$

The hypothesis that H is a normal subgroup of G tells us that $c^{-1}h_1^{-1}c$ is in H; therefore, $(c^{-1}h_1^{-1}c)h_2^{-1}h \in H$, since H is a group. We have shown that bdh is equal to ac times an element of H; and therefore, $bdh \in acH$.

5. **(7 points) Give an example of a subgroup** H **of a group** G **and elements** a**,** b**,** c**, and** d **of** G **with** $aH = bH$ **and** $cH = dH$**, but** $acH \neq bdH$**.**

Let $G = D_3$, $H = <\sigma>$, $a = \text{id}$, $b = \sigma$, $c = \rho$, and $d = \sigma\rho^2$. We have

$$aH = bH = \{\text{id}, \sigma\} \quad \text{and} \quad cH = dH = \{\rho, \sigma\rho^2\};$$

however,

$$acH = \text{id}\rho H = \rho H = \{\rho, \sigma\rho^2\} \quad \text{and} \quad bdH = \sigma\sigma\rho^2 H = \rho^2 H = \{\rho^2, \sigma\rho\}.$$

Thus, $acH \neq bdH$.

6. **(7 points) Write the addition table for** $\frac{\mathbb{Z}}{4\mathbb{Z}}$**.**

	$0+4\mathbb{Z}$	$1+4\mathbb{Z}$	$2+4\mathbb{Z}$	$3+4\mathbb{Z}$
$0+4\mathbb{Z}$	$0+4\mathbb{Z}$	$1+4\mathbb{Z}$	$2+4\mathbb{Z}$	$3+4\mathbb{Z}$
$1+4\mathbb{Z}$	$1+4\mathbb{Z}$	$2+4\mathbb{Z}$	$3+4\mathbb{Z}$	$0+4\mathbb{Z}$
$2+4\mathbb{Z}$	$2+4\mathbb{Z}$	$3+4\mathbb{Z}$	$0+4\mathbb{Z}$	$1+4\mathbb{Z}$
$3+4\mathbb{Z}$	$3+4\mathbb{Z}$	$0+4\mathbb{Z}$	$1+4\mathbb{Z}$	$2+4\mathbb{Z}$

7. **(7 points) Let** $(S, *)$ **be the group** $(\mathbb{R} \setminus \{-1\}, *)$ **with** $a * b = ab + a + b$ **for all** a **and** b **in** S**. Define the function** $\varphi\colon (\mathbb{R} \setminus \{0\}, \times) \to (S, *)$ **by** $\varphi(r) = r - 1$**. Prove that** φ **is a group homomorphism.**

We see that φ is a function from $\mathbb{R} \setminus \{0\}$ to $\mathbb{R} \setminus \{-1\}$. We also see that if a and b are in $\mathbb{R} \setminus \{0\}$, then

$$\begin{aligned}\varphi(a)*\varphi(b) = (a-1)*(b-1) = (a-1)(b-1)+(a-1)+(b-1) &= ab-a-b+1+a-1+b-1\\ &= ab - 1 = \varphi(a \times b).\end{aligned}$$