Math 546, Exam 3, Fall 2011

Write everything on the blank paper provided.

You should KEEP this piece of paper.

If possible: turn the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam. The exam is worth 50 points. There are **8** problems.

Write **coherently** in **complete sentences**. **No Calculators or Cell phones.**

1. (7 points) Define *normal subgroup*. Use complete sentences. Write everything that is necessary for your definition to make sense, but nothing extra.

2. (7 points) Define *group homomorphism*. Use complete sentences. Write everything that is necessary for your definition to make sense, but nothing extra.

3. (6 points) Let ℓ, m, and n be fixed positive integers and let H be the subgroup
$$H = \{am + bn \mid a, b \in \mathbb{Z}\}$$
of $\mathbb{Z}$. (I believe that H is a subgroup. I do not need to see a proof.) Suppose that H is also equal to $\{c\ell \mid c \in \mathbb{Z}\}$. Prove that ℓ is the greatest common divisor of n and m.

4. (6 points) Let $S = \mathbb{R} \setminus \{-2\}$. Define an operation $*$ on S by $a * b = ab + 2a + 2b + 2$. I believe that $(S, *)$ is a group. I want you to exhibit a group isomorphism from $(\mathbb{R} \setminus \{0\}, \times)$ to $(S, *)$. Prove that your candidate is a group isomorphism.

5. (6 points) Let X and Y be sets. Suppose that $f\colon X \to Y$ and $g\colon Y \to X$ are functions. Suppose further that $(g \circ f)(x) = x$ for all x in X.
   (a) Does f have to be one-to-one? If yes, prove it. If no, give an example.
   (b) Does f have to be onto? If yes, prove it. If no, give an example.

6. (6 points) Let $G = <g>$ be a cyclic group of order 48. Draw the lattice of subgroups of G.

7. (6 points) Prove that every infinite cyclic group is isomorphic to $(\mathbb{Z}, +)$.

8. (6 points) Consider the function $\varphi\colon (\mathbb{R}^2, +) \to (\mathbb{R}, +)$ which is given by $\varphi\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = a + b$. Is f a group homomorphism? If yes, prove it **and identify the kernel and image of** φ. If no, **give a counterexample.**